STA 130A: Mathematical Statistics: Brief Course

Practice Final Exam A

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Instructions: This practice final exam is **closed-book**, except for the permitted note sheet described below. You may bring a pen or pencil, three letter-sized sheets of *hand-written* notes (both sides), and a *non-graphing* calculator. No other materials (e.g., textbooks) are allowed. For a timed attempt, allow 120 minutes. Problems 1–6 total **180 points**; Problem 7 offers *up to 10 bonus points*. Once you receive this exam, please confirm that you have all 11 pages.

- Make sure to clearly write your name and student ID above.
- Present answers succinctly, but include all relevant steps for full credit. Partial credit is possible only if your reasoning is clearly shown and traceable by the grader.
- If necessary, round numerical answers to three decimal places; fractions, logarithms, and exponentials may be left unevaluated.
- Complete Problems 1–6 before attempting the optional bonus questions in Problem 7.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Problem 7 (bonus)	
Total	

Problem 1 (60 points in total). Fundamentals

Answer each part separately.

(a) (5 points) A draw is made from jar A or B, with $P(B) = 1/3$. Let R denote a red draw. Given $P(R \mid A) = 1/4$ and $P(R \mid B) = 3/4$, find $P(B \mid R)$.

(b) (5 points) Independent trials succeed with probability 0.4. Let N be the trial number until the first success. Find $P(N - 4 = k \mid N > 4)$ for $k = 1, 2, \ldots$.

(c) (5 points) Using $\Phi^{-1}(0.90) \approx 1.282$, give the 90th percentile of $N(50, 5^2)$ to two decimal places.

(d) (5 points) Given $\sigma_X = 2$, $\sigma_Y = 3$, and $\mathrm{Corr}(X, Y) = -1/2$, find $\mathrm{Var}(X - Y)$.

(e) (5 points) The MGF of X is $M_X(t) = (3 + e^{2t})/4$. Find $\mathrm{Var}(X)$.

(f) (5 points) Let D be a nonnegative delay in minutes with $\mathbb{E}[D] = 3$. Using only this information, derive the best possible upper bound on $P(D \geq 24)$.

(g) (5 points) Independent readings each have standard deviation 4. Find the smallest sample size for which their average has standard error at most 1.25.

(h) (5 points) For nine independent $N(12, 3^2)$ readings, calculate $P(11.25 < \bar{X} < 13.40)$ numerically.

(i) (5 points) For 13 i.i.d. $N(\mu, \sigma^2)$ readings with $\sigma > 0$, find $P(\bar{X} > \mu + \sigma/\sqrt{13})$. Would knowing the sample variance change your answer? Briefly explain.

(j) (5 points) For i.i.d. $N(\mu, \sigma^2)$ readings with known $\sigma > 0$, let I_n be the interval $\bar{X}_n \pm 1.96\sigma/\sqrt{n}$. Express $P(X_{n+1} \in I_n)$ in terms of Φ.

(k) (5 points) For observations $x_1, \ldots, x_n$, the sample mean is $\bar{x} = 3$ and $n^{-1}\sum_{i=1}^{n} x_i^2 = 11$. Use $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ to find the method-of-moments estimate of the population variance.

(l) (5 points) Four independent Bernoulli(p) trials yield $(1, 1, 0, 1)$, where 1 denotes success. For these observations, compute the likelihood ratio $L(3/4)/L(1/2)$.

Problem 2 (20 points in total). An error count and an audit flag

The table gives the joint PMF of the error count X and flag Y (1 if flagged) for a randomly selected file.

	$X = 0$	$X = 1$	$X = 2$
$Y = 0$	3/12	2/12	1/12
$Y = 1$	1/12	2/12	3/12

(a) (10 points) Find the marginal PMF of the error count X.

(b) (10 points) In a new audit, $P(Y = 1) = 3/4$ and the conditional PMF of X in each flag group is unchanged. Find the new $\mathbb{E}[X]$ and explain why it differs from the original mean of 1.

Problem 3 (20 points in total). Continuous distributions

The parts use separate models. Give each PDF on and off its support.

(a) (10 points) The joint PDF is $2/3$ on $0 < x < 1$, $x < y < 2$, and zero elsewhere. Given $f_X(x) = \frac{2}{3}(2-x)$ for $0 < x < 1$, find $f_{Y|X}(y \mid x)$ for such x.

(b) (10 points) Let $W \sim \text{Uniform}(-1, 2)$ and $Z = |W|$. Derive the piecewise PDF of Z. For each nonzero piece, identify the values of W that map to z.

Problem 4 (20 points in total). Sampling distributions and confidence intervals

(a) (10 points) In each of two studies, S is the sum of 64 i.i.d. observations, each with mean 3 and SD 2. Study A records continuous measurements. Study B records counts on consecutive integers. For each study, find the cutoff z in the normal approximation $P(S \geq 208) \approx 1 - \Phi(z)$, using a continuity correction where appropriate.

(b) (10 points) Nine i.i.d. normal readings give $\bar{x} = 11.4$, $s = 2$. Use $t_8^* \approx 2.306$ and $z^* \approx 1.960$ for 95% intervals.

(i) With σ unknown, construct the exact equal-tailed 95% confidence interval for μ.

(ii) In a new study, $\bar{X}_n$ and $\bar{Y}_m$ are means of independent random samples from $N(\mu, 2^2)$. For given n, m, find the smallest half-width $w_{n,m}$ such that $P(|\bar{Y}_m - \bar{X}_n| \leq w_{n,m}) \geq 0.95$. Then find the least n for which $w_{n,m} \leq 1$ for some finite m.

Problem 5 (20 points in total). Combining two instruments

Calibrated readings $X \sim N(\mu, 1)$ and $Y \sim N(\mu, 4)$ are independent. Their log likelihood is

$$\ell(\mu) = C - \frac{(x-\mu)^2}{2} - \frac{(y-\mu)^2}{8}, \qquad C \text{ constant in } \mu.$$

(a) (10 points) Derive the maximum likelihood estimator of μ and justify that it maximizes the likelihood.

(b) (10 points) In m independent pairs, only the mean of Y_i changes, to $\mu + \delta$ for a fixed offset δ. Apply the rule from (a) to each pair as if $\delta = 0$. Let A_m average those estimates and let $\bar{X} = m^{-1} \sum_{i=1}^{m} X_i$.

(i) Find the MSEs of A_m and $\bar{X}$ as estimators of μ.

(ii) Suppose $|\delta| \le d$ with $d > 0$. In terms of m and d, determine when each estimator has the smaller worst-case MSE and when they tie.

Problem 6 (40 points in total). An anonymous survey

Each respondent has a trait indicator $T \sim \text{Bernoulli}(p)$, $0 < p < 1$. With probability $2/3$, the respondent answers truthfully (yes if $T = 1$). Otherwise, a fair coin decides the answer (yes on heads). The rule choice and coin are independent of each other and of T. Respondents and their randomizations are independent across people. Only the yes indicator Y is observed.

(a) (10 points) Find the probability $q = P(Y = 1)$ in terms of p.

(b) (10 points) Find the conditional mean $\mathbb{E}[Y \mid T]$ and conditional variance $\text{Var}(Y \mid T)$ for each trait group $T = 0, 1$.

(c) (10 points) For n independent responses, consider the following two estimators of p.

(i) Use $\bar{Y}$ to obtain the unconstrained moment estimator $\hat{p}$. Find its bias and variance in terms of p and n.

(ii) Let $\tilde{p}$ be the point in $[0, 1]$ closest to $\hat{p}$. Does $\tilde{p}$ have strictly smaller MSE than $\hat{p}$ for every $0 < p < 1$ and finite $n \geq 1$? Justify.

(d) (10 points) Compare $\hat{p}$ from (c)(i) with the proportion from 64 independent truthful responses. Find the least n for which $\hat{p}$ has no larger standard error for *every* $p \in [1/4, 3/4]$. Justify. You may use $\text{Var}(\hat{p}) = \{p(1-p) + b\}/n$ with your constant b from (c)(i).

Problem 7 (bonus)

(a) (5 points) Let $X \sim \text{Uniform}(-1, 1)$ and $Y = (X - a)^2$. Find all $a \in \mathbb{R}$ for which X and Y are uncorrelated. For each such a, decide whether they are independent and justify your answer.

(b) (5 points) Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p), $0 < p < 1$, $n \geq 2$. The pair product $X_1 X_2$ has mean p^2. Construct an estimator of p^2 using only the total $K = \sum_i X_i$, and show that it is both unbiased and consistent.

Appendix. Standard normal probabilities

For $Z \sim N(0,1)$ the entries give $\Phi(z) = P(Z \le z)$. For $0 \le z \le 2.49$, use the row for the units and tenths and the column for the hundredths. For negative values, $\Phi(-z) = 1 - \Phi(z)$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936