

STA 130A: Mathematical Statistics: Brief Course

Practice midterm exam 1

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Name: _____ Student ID: _____

Instructions: This practice midterm is **closed-book**, except for the permitted note sheet described below. You may bring a pen or pencil, one letter-sized sheet of *hand-written* notes (both sides), and a *non-graphing* calculator. No other materials (e.g., textbooks) are allowed. You have 50 minutes to complete all problems. The **total score is 120 points**, with *up to 10 bonus points available*. Once you receive this exam, please confirm that you have all 7 pages, including the standard normal table in the appendix.

- Make sure to clearly write your name and student ID above.
- Present answers succinctly, but include all relevant steps for full credit. Partial credit is possible only if your reasoning is clearly shown and traceable by the grader.
- If necessary, round numerical answers to three decimal places. Exact expressions, including fractions and counting expressions, are acceptable.
- Bonus problems can be more challenging; consider attempting them after you finish the main problems.

Problem	Score
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Bonus	
Total	

Problem 1 (40 points in total). Warm-up

Treat each item separately. Most items require only one or two lines of calculation or justification.

- (a) **(5 points)** Let $N \sim \text{Poisson}(2)$. Calculate $P(N = 3)/P(N = 2)$.
(Hint: The Poisson PMF is $P(N = k) = e^{-2}2^k/k!$ for $k = 0, 1, 2, \dots$)
- (b) **(5 points)** From seven distinct objects, how many three-element subsets contain one specified object?
- (c) **(5 points)** Let $N \sim \text{Binomial}(5, 1/4)$. Calculate $P(N \geq 1)$.
- (d) **(5 points)** Events E and F satisfy $E \subseteq F$. Could $P(E) = 0.4$ and $P(F) = 0.3$? Explain in one sentence.
- (e) **(5 points)** Events A and B are independent, with $P(A) = 1/4$ and $P(B) = 2/3$. Calculate $P(A^c \cap B)$.
- (f) **(5 points)** Suppose $P(C) = 2/5$, $P(D | C) = 1/2$, and $P(D | C^c) = 1/6$. Calculate $P(D)$.
- (g) **(5 points)** Suppose $\text{Var}(X) = 3$. Calculate $\text{Var}(2 - 3X)$.
- (h) **(5 points)** Let $Y \sim N(20, 5^2)$. Approximate the value of c satisfying $P(Y > c) = 0.10$, using the closest entry in the standard normal table in the appendix.

Problem 2 (25 points in total). Constructing discrete distributions

Let U, V be independent random variables, each uniform on $\{1, 2, 3\}$. Define

$$M = \max\{U, V\}, \quad L = \min\{U, V\}.$$

(a) (7 points) Write the PMF $p_M(m)$ as a piecewise formula.

(b) (8 points) Calculate $P(U < V \mid U + V = 4)$.

(c) (10 points) Let $S = U + V$. Write the conditional PMF of S given $L = 1$ as a piecewise formula.

Problem 3 (25 points in total). A distribution specified by its CDF

The CDF of a random variable X is

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-x/2}, & x \geq 0. \end{cases}$$

(a) (7 points) Write a PDF $f_X(x)$ as a piecewise formula.

(b) (8 points) Calculate the value of q satisfying $P(X \leq q) = 3/4$.

(c) (10 points) Using the given CDF, show that, for every $s, t \geq 0$,

$$P(X > s + t \mid X > s) = P(X > t).$$

Interpret this identity in one sentence when X represents a waiting time.

Problem 4 (30 points in total). Correcting placements

Documents 1, 2, 3, 4 are placed into envelopes 1, 2, 3, 4, one document per envelope, with all $4!$ placements equally likely. For $i = 1, 2, 3, 4$, let $I_i = 1$ if document i is correctly placed in envelope i , and $I_i = 0$ otherwise. The number of correctly placed documents is

$$X = \sum_{i=1}^4 I_i.$$

(a) (10 points) How many placements have exactly two correctly placed documents? Explain your count.

(b) (10 points) Calculate $\mathbb{E}[X]$ and $\text{Var}(X)$. Show your reasoning.

(c) (10 points) Keep all correctly placed documents fixed. Redistribute the others among the remaining envelopes, one per envelope, with all arrangements equally likely, including the current arrangement. If all four documents are already in correct envelopes, nothing changes.

Let Y be the total number of correctly placed documents afterward. For each possible initial count x , calculate the expected final count, $\mathbb{E}[Y \mid X = x]$. Write your answer as a piecewise formula and justify.

Optional bonuses (5 points each)

Complete the required problems first. A brief, well-supported argument is sufficient.

Bonus 1 (5 points). Random shares

Let X_1, X_2, X_3 be independent, identically distributed random variables with $P(X_i > 0) = 1$. Calculate

$$\mathbb{E} \left[\frac{X_1}{X_1 + X_2 + X_3} \right]$$

without specifying their common distribution. Briefly justify your answer.

Bonus 2 (5 points). Dependence and distance

X and Y are each uniform on $\{0, 1, 2\}$, but they need not be independent. What is the largest possible value of $\mathbb{E}[|X - Y|]$? Write a joint PMF, as a table or piecewise formula, attaining it, and give a brief argument that no larger value is possible.

Appendix. Standard normal table

For $Z \sim N(0, 1)$, this table gives $\Phi(z) = P(Z \leq z)$. The row gives the units and tenths of z , and the column gives the hundredths. For negative z , use $\Phi(-z) = 1 - \Phi(z)$. For a requested quantile, use the closest table entry; interpolation is not required.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990