

STA 130A: Mathematical Statistics: Brief Course

Practice midterm exam 1 solution

Instructor: Dogyoon Song

Problem 1 (40 points). Warm-up

(a) (5 points) $\frac{P(N=3)}{P(N=2)} = \frac{e^{-2}2^3/3!}{e^{-2}2^2/2!} = \boxed{\frac{2}{3}}.$

(b) (5 points) Include the specified object, then choose two of the remaining six: $\boxed{\binom{6}{2} = 15}.$

(c) (5 points) $P(N \geq 1) = 1 - P(N=0) = 1 - \left(\frac{3}{4}\right)^5 = \boxed{\frac{781}{1024}}.$

(d) (5 points) **No:** $E \subseteq F$ requires $P(E) \leq P(F)$, contradicting $0.4 > 0.3$.

(e) (5 points) Independence gives $P(A^c \cap B) = (1 - P(A))P(B) = \frac{3}{4} \frac{2}{3} = \boxed{\frac{1}{2}}.$

(f) (5 points) By total probability, $P(D) = \frac{2}{5} \frac{1}{2} + \frac{3}{5} \frac{1}{6} = \boxed{\frac{3}{10}}.$

(g) (5 points) $\text{Var}(2 - 3X) = (-3)^2 \text{Var}(X) = 9(3) = \boxed{27}.$

(h) (5 points) The standardized cutoff satisfies $\Phi((c - 20)/5) = 0.90$. The closest table entry is $\Phi(1.28) = 0.8997$, so $\boxed{c \approx 20 + 5(1.28) = 26.4}.$

Problem 2 (25 points). Constructing discrete distributions

All nine ordered pairs (U, V) are equally likely.

- (a) (7 points) There are $m^2 - (m-1)^2 = 2m - 1$ pairs with maximum m . Hence

$$p_M(m) = \begin{cases} \frac{2m-1}{9}, & m \in \{1, 2, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

- (b) (8 points) Given $U+V=4$, the equally likely pairs are $(1, 3), (2, 2), (3, 1)$. Only the first satisfies $U < V$, so $P(U < V \mid U+V=4) = 1/3$.

- (c) (10 points) Given $L=1$, the five equally likely pairs are $(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)$, with sums 2, 3, 3, 4, 4. Therefore

$$p_{S|L}(s \mid 1) = \begin{cases} \frac{1}{5}, & s = 2, \\ \frac{2}{5}, & s \in \{3, 4\}, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 3 (25 points). A distribution specified by its CDF

- (a) (7 points) Differentiating the CDF away from zero gives a valid PDF:

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The PDF's value at $x=0$ does not affect the distribution; using the positive formula for $x \geq 0$ is also valid.

- (b) (8 points)

$$1 - e^{-q/2} = \frac{3}{4} \implies e^{-q/2} = \frac{1}{4} \implies q = 2 \ln 4 \approx 2.773.$$

- (c) (10 points) For $s, t \geq 0$, $\{X > s+t\} \subseteq \{X > s\}$, so

$$P(X > s+t \mid X > s) = \frac{1 - F_X(s+t)}{1 - F_X(s)} = \frac{e^{-(s+t)/2}}{e^{-s/2}} = e^{-t/2} = P(X > t).$$

Given that the wait has lasted s time units, the remaining waiting time has the same distribution as the original waiting time.

Problem 4 (30 points). Correcting placements

(a) (10 points) Choose which two documents are correctly placed; the other two must be swapped. Thus

there are $\boxed{\binom{4}{2} = 6}$ placements.

(b) (10 points) For each i , $\mathbb{E}[I_i] = 3!/4! = 1/4$; for $i \neq j$, $\mathbb{E}[I_i I_j] = 2!/4! = 1/12$. Using $I_i^2 = I_i$,

$$\begin{aligned}\mathbb{E}[X] &= 4 \left(\frac{1}{4} \right) = \boxed{1}, \\ \mathbb{E}[X^2] &= \sum_{i=1}^4 \mathbb{E}[I_i] + 2 \sum_{i < j} \mathbb{E}[I_i I_j] = 1 + 2 \binom{4}{2} \frac{1}{12} = 2.\end{aligned}$$

Consequently, $\boxed{\text{Var}(X) = 2 - 1^2 = 1}$.

(c) (10 points) The possible initial counts are 0, 1, 2, 4: if three documents are correctly placed, the fourth must also be correct. For $x < 4$, each of the $m = 4 - x$ redistributed documents has probability $1/m$ of being placed correctly, so the expected number newly correct is $m(1/m) = 1$. Thus

$$\mathbb{E}[Y \mid X = x] = \begin{cases} x + 1, & x \in \{0, 1, 2\}, \\ 4, & x = 4. \end{cases}$$

Optional bonuses (5 points each)**Bonus 1. Random shares**

Let $R_i = X_i / (X_1 + X_2 + X_3)$. The shares are bounded, so their expectations exist. Symmetry gives equal expectations, while $R_1 + R_2 + R_3 = 1$. Therefore

$$3\mathbb{E}[R_1] = 1, \quad \boxed{\mathbb{E}\left[\frac{X_1}{X_1 + X_2 + X_3}\right] = \frac{1}{3}}.$$

This argument does not require $\mathbb{E}[X_i]$ to be finite.

Bonus 2. Dependence and distance

The pointwise inequality $|X - Y| \leq |X - 1| + |Y - 1|$ gives

$$\mathbb{E}[|X - Y|] \leq \mathbb{E}[|X - 1|] + \mathbb{E}[|Y - 1|] = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

The bound is attained by the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{3}, & (x,y) \in \{(0,2), (1,1), (2,0)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Both marginals are uniform on $\{0, 1, 2\}$, and this PMF gives $\mathbb{E}[|X - Y|] = (2 + 0 + 2)/3 = \boxed{4/3}$, the largest possible value.