

STA 130A: Mathematical Statistics: Brief Course

Midterm exam 1 solution

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Problem 1 (40 points). Warm-up

(a) (5 points) The first two trials must fail and the third must succeed:

$$P(N = 3) = \left(\frac{3}{4}\right)^2 \frac{1}{4} = \boxed{\frac{9}{64}}.$$

(b) (5 points) **No:** disjointness gives $P(A \cap B) = 0$, whereas independence would require $P(A \cap B) = P(A)P(B) = 1/12 > 0$.

(c) (5 points) With the last digit fixed at 6, there are five choices for the first digit and four for the second:

$$\boxed{5 \cdot 4 = 20}.$$

(d) (5 points) By linearity,

$$\mathbb{E}[Y] = \mathbb{E}[Y - X] + \mathbb{E}[X] = 3 + 2 = \boxed{5}.$$

(e) (5 points) Each $1 - I_i$ is Bernoulli(3/4), and these variables are independent. Thus

$$6 - \sum_{i=1}^6 I_i = \sum_{i=1}^6 (1 - I_i) \sim \boxed{\text{Binomial}(6, 3/4)}.$$

(f) (5 points) **No:** a CDF is nondecreasing, but the proposed values give $F(0) > F(1)$ although $0 < 1$.

(g) (5 points) Since $C \cup D$ is the disjoint union of D and $C \cap D^c$,

$$P(C \cap D^c) = P(C \cup D) - P(D) = \frac{9}{10} - \frac{1}{2} = \boxed{\frac{2}{5}}.$$

(h) (5 points) The joint probability is $P(A \cap B) = (1/3)(3/4) = 1/4$. By total probability and Bayes' rule,

$$P(B) = \frac{1}{3} \frac{3}{4} + \frac{2}{3} \frac{3}{8} = \frac{1}{2}, \quad P(A | B) = \frac{(1/3)(3/4)}{(1/3)(3/4) + (2/3)(3/8)} = \boxed{\frac{1}{2}}.$$

Problem 2 (25 points). Joint distributions and moments**(a) (7 points)** Summing the rows of the joint PMF gives

$$p_X(x) = \begin{cases} \frac{1}{4}, & x = 0, \\ \frac{1}{2}, & x = 1, \\ \frac{1}{4}, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (6 points) The variables are **not independent**. For example,

$$p_{X,Y}(0,0) = \frac{3}{16} \neq p_X(0)p_Y(0) = \frac{1}{4} \left(\frac{3+2+3}{16} \right) = \frac{1}{8}.$$

(c) (6 points) Since $P(Y = 1) = (1 + 6 + 1)/16 = 1/2$,

$$\mathbb{E}[X^2 \mid Y = 1] = \frac{0^2(1/16) + 1^2(6/16) + 2^2(1/16)}{8/16} = \boxed{\frac{5}{4}}.$$

(d) (6 points) Let $S = X + Y$. Grouping the joint-PMF entries by their sum gives

$$\begin{aligned} \mathbb{E}[S] &= \frac{0(3) + 1(1+2) + 2(6+3) + 3(1)}{16} = \frac{3}{2}, \\ \mathbb{E}[S^2] &= \frac{0^2(3) + 1^2(1+2) + 2^2(6+3) + 3^2(1)}{16} = 3. \end{aligned}$$

Therefore,

$$\text{Var}(X + Y) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}.$$

Alternatively, the table gives $\mathbb{E}[X] = 1$, $\mathbb{E}[Y] = 1/2$, $\mathbb{E}[XY] = 1/2$, $\text{Var}(X) = 1/2$, and $\text{Var}(Y) = 1/4$. The cross term $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ is zero, so the variances add even though the variables are dependent.

Problem 3 (25 points). Densities and functions of random variables**(a) (6 points)** Normalization requires

$$1 = \int_0^2 c(1+x) dx = c \left[x + \frac{x^2}{2} \right]_0^2 = 4c, \quad \boxed{c = \frac{1}{4}}.$$

(b) (10 points) Only $1 < X < 2$ contributes to the expectation:

$$\mathbb{E}[Q] = \int_1^2 (x-1)c(1+x) dx = c \int_1^2 (x^2-1) dx = c \left[\frac{x^3}{3} - x \right]_1^2 = \boxed{\frac{4c}{3}}.$$

(c) (9 points) Standardizing with standard deviation 2 gives

$$\begin{aligned} P(8 \leq Y \leq 13) &= P\left(-1 \leq \frac{Y-10}{2} \leq 1.5\right) \\ &= \Phi(1.5) - \Phi(-1) = 0.9332 - (1 - 0.8413) \\ &= 0.7745 \approx \boxed{0.775}. \end{aligned}$$

Problem 4 (30 points). Assigning tasks

- (a) (10 points) Choose one of the two experienced workers and two of the four trainees, then assign the three selected workers to the three distinct tasks:

$$\binom{2}{1} \binom{4}{2} 3! = 2 \cdot 6 \cdot 6 = 72.$$

Equivalently, choose the experienced worker's task and then fill the tasks in order: $3 \cdot 2 \cdot 4 \cdot 3 = 72$.

- (b) (10 points) An experienced worker finishes by one hour with probability 1. A trainee's completion time $U \sim \text{Uniform}(0, 2)$ satisfies $P(U \leq 1) = 1/2$. Hence

$$\mathbb{E}[Y \mid X = x] = x + \frac{3-x}{2} = \frac{3+x}{2}.$$

The stated staffing PMF gives $\mathbb{E}[X] = 0(1/4) + 1(1/2) + 2(1/4) = 1$, so

$$\mathbb{E}[Y] = \mathbb{E}\left[\frac{3+X}{2}\right] = 2.$$

- (c) (10 points) Let $H = \{T \leq 1/2\}$. Experienced workers finish before this deadline; each trainee does so with probability $(1/2)/2 = 1/4$. Conditional independence therefore gives

$$P(H \mid X = x) = \left(\frac{1}{4}\right)^{3-x}, \quad x = 0, 1, 2.$$

Using the stated PMF,

$$P(H) = \frac{1}{4} \left(\frac{1}{4}\right)^3 + \frac{1}{2} \left(\frac{1}{4}\right)^2 + \frac{1}{4} \left(\frac{1}{4}\right) = \frac{25}{256}.$$

Bayes' rule yields

$$P(X = 2 \mid H) = \frac{(1/4)(1/4)}{25/256} = \frac{16}{25} = 0.640.$$

Observing all three tasks completed within half an hour raises the probability that both experienced workers were assigned from $1/4 = 0.250$ to $16/25 = 0.640$.

Optional bonuses (5 points each)**Bonus 1. Integer values and variance**

Every integer lies at least $1/2$ from $3/2$. Thus

$$\text{Var}(X) = \mathbb{E} \left[\left(X - \frac{3}{2} \right)^2 \right] \geq \frac{1}{4}.$$

This bound is attained by

$$p_X(x) = \begin{cases} \frac{1}{2}, & x \in \{1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$$

Its mean is $(1 + 2)/2 = 3/2$, and its squared deviation from the mean is always $1/4$. Therefore the smallest possible variance is $\boxed{1/4}$.

Bonus 2. Three equally likely labels

No such rule exists. Imagine generating three independent fair tosses in advance. There are eight equally likely complete sequences. A rule that stops early still assigns a definite label to each complete sequence: every continuation of a stopping prefix receives the same label. Each label consequently has probability $k/8$ for an integer k . Since $1/3$ is not an integer multiple of $1/8$, equal probabilities $1/3$ are impossible.