STA 130A: Mathematical Statistics: Brief Course

Midterm exam 2

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Name: _____ **Student ID:** _____

Instructions: This midterm exam is **closed-book**, except for the permitted note sheet described below. You may bring a pen or pencil, one letter-sized sheet of *hand-written* notes (both sides), and a *non-graphing* calculator. No other materials (e.g., textbooks) are allowed. You have 50 minutes for the exam. The **total score is 120 points**, with *up to 10 bonus points available*. Once you receive this exam, please confirm that you have all 6 pages.

- Make sure to clearly write your name and student ID above.
- Present answers succinctly, but include all relevant steps for full credit. Partial credit is possible only if your reasoning is clearly shown and traceable by the grader.
- If necessary, round numerical answers to three decimal places. Exact expressions, including fractions and factored expressions, are acceptable.
- Bonus problems can be more challenging; consider attempting them after you finish the main problems.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5 (bonus)	
Total	

Problem 1 (40 points in total). Conceptual checks

Treat each item separately. One or two lines of calculation or justification suffice.

(a) (8 points) Suppose $\mathrm{Var}(X) = 1$, $\mathrm{Var}(Y) = 4$, and $\mathrm{Corr}(X, Y) = -1/2$. Calculate $\mathrm{Var}(X + Y)$.

(b) (8 points) The joint PDF of U, V is $f_{U,V}(u, v) = 1$ for $0 < u < 1$ and $0 < v < 1$, and zero elsewhere. Calculate $P(U + V < 1/2)$.

(c) (8 points) Let $Y = 3X - 1$. Calculate $\mathrm{Var}(Y \mid X)$.

(d) (8 points) The MGF of X is $M_X(t) = (1 - t)^{-2}$ for $t < 1$. Use M_X to calculate $\mathrm{Var}(X)$.

(e) (8 points) Suppose $W \geq 0$ and $\mathbb{E}[W] = 3$, with no other information about its distribution available. Report the best upper bound on $P(W \geq 12)$ you can derive. A proof of optimality is not required.

Problem 2 (30 points in total). Continuous distributions

The three parts use **separate probability models**. For each requested PDF, specify where it is nonzero and give its value elsewhere.

(a) (10 points) The joint PDF of X, Y is

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & 0 < x < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Write the marginal PDF $f_Y(y)$.

(b) (10 points) A different pair U, V has joint PDF

$$f_{U,V}(u,v) = \begin{cases} 3v, & 0 < u < v < 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < v < 1$, write the conditional PDF $f_{U|V}(u \mid v)$ as a function of u.

(c) (10 points) Separately, let $W \sim \text{Uniform}(0,1)$ and $Z = 1/(1+W)$. Write the PDF $f_Z(z)$.

Problem 3 (20 points in total). Independent sums and correlation

A random variable R has PMF

$$p_R(r) = \begin{cases} 3/4, & r = 0, \\ 1/4, & r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let R_1, R_2, R_3 be independent random variables with this same PMF, and define

$$S = 1 + R_1 + R_2 + R_3.$$

(a) (10 points) Starting from the given PMF, calculate $M_S(t)$ for all real t.
Hint: You may begin by computing $M_R(t)$ and then use independence.

(b) (10 points) Calculate $\text{Corr}(R_1, S)$. Show your reasoning.

Problem 4 (30 points in total). Planning deliveries

A company sends two packages in each dispatch. It chooses route A or route B with equal probability, and sends both packages using that route. Let G be the chosen route and T_1, T_2 the delivery times, in minutes. **Conditional on** G, these times are independent and have the same distribution. For $j \in \{1, 2\}$,

	$G = \text{A}$	$G = \text{B}$
$\mathbb{E}[T_j \mid G]$	10	14
$\text{Var}(T_j \mid G)$	9	4
$P(T_j \leq 12 \mid G)$	3/4	1/4

Variances are in squared minutes. No distributional forms are assumed beyond this information.

(a) (10 points) Calculate $\text{Var}(T_1)$.

(b) (10 points) A package is on time if it arrives within 12 minutes. For $j \in \{1, 2\}$, let $I_j = 1$ if package j is on time, and $I_j = 0$ otherwise. Calculate $\text{Cov}(I_1, I_2)$.
Hint: You may begin by computing $P(T_j \leq 12)$ for $j \in \{1, 2\}$.

(c) (10 points) For a separate study using **route B only**, the company will record n independent delivery times $U_1, \ldots, U_n$, each with the route-B distribution above. Let $\overline{U}_n = n^{-1} \sum_{i=1}^{n} U_i$ be their average. Using the given mean and variance, derive as tight a probability bound as possible using the inequalities covered in class. Find the **smallest positive integer** n for which **your bound** guarantees

$$P\big(|\overline{U}_n - 14| < 2\big) \geq 0.95.$$

Problem 5 (10 bonus points in total). Optional bonuses

Attempt these optional problems after Problems 1–4. For each, give a yes/no answer supported by a mathematical argument; a conclusion alone is not sufficient.

(a) (5 bonus points) Can a discrete random variable X satisfy all three conditions below?

$$\mathbb{E}[X] = 0, \qquad \mathrm{Var}(X) = 1, \qquad P(X \geq 2) = \frac{1}{4}.$$

Give a PMF and verify the conditions, or prove that no such variable exists.

(b) (5 bonus points) Let X and Y each have a Bernoulli(1/2) distribution, without assuming independence. Can $\mathrm{Cov}(X, Y) = 0$ hold while X and Y are dependent? Justify your answer.