# STA 130A: Mathematical Statistics: Brief Course

## Practice midterm 2 – Set A

Instructor: Dogyoon Song

**Name:** _____ **Student ID:** _____

**Instructions:** This practice midterm is **closed-book**, except for the permitted note sheet described below. You may bring a pen or pencil, one letter-sized sheet of *hand-written* notes (both sides), and a *non-graphing* calculator. No other materials (e.g., textbooks) are allowed. For a timed attempt, allow 50 minutes. The **total score is 120 points**, with *up to 10 bonus points available.* Please confirm that you have all 6 pages.

- Make sure to clearly write your name and student ID above.
- Present answers succinctly, but include all relevant steps for full credit. Partial credit is possible only if your reasoning is clearly shown and traceable by the grader.
- If necessary, round numerical answers to three decimal places. Exact expressions, including fractions and factored expressions, are acceptable.
- Bonus problems can be more challenging; consider attempting them after you finish the main problems.

| **Problem** | **Score** |
|---|---|
| Problem 1 | |
| Problem 2 | |
| Problem 3 | |
| Problem 4 | |
| Problem 5 (bonus) | |
| **Total** | |

## Problem 1 (30 points in total). Conceptual checks

Treat each item separately. One or two lines of calculation or justification suffice.

**(a) (5 points)** Let $U \sim \text{Uniform}(0, 1)$ and $Z = 3 - 2U$. Write the PDF $f_Z(z)$, including where it is zero.

**(b) (5 points)** Suppose $X, Y$ have finite, positive variances. Does $\text{Corr}(X, Y) = 0$ necessarily imply independence? Explain in one sentence; a constructed example is not required.

**(c) (5 points)** Suppose $\text{Var}(X) = 1$, $\text{Var}(Y) = 4$, and $\text{Cov}(X, Y) = 1$. Calculate $\text{Var}(2X - Y)$.

**(d) (5 points)** Let $G$ be a discrete group label and let $Y$ have finite variance. Must $\text{Var}(Y \mid G = g) \leq \text{Var}(Y)$ hold for every $g$ with $P(G = g) > 0$? Explain in one sentence.

**(e) (5 points)** The MGF of $X$ is $M_X(t) = (1 - 2t)^{-1}$ for $t < 1/2$. Use its derivatives at 0 to calculate $\mathbb{E}[X^2]$.

**(f) (5 points)** Suppose $\mathbb{E}[X^2] = 9$. Apply Markov's inequality to $X^2$ to obtain a numerical upper bound for $P(|X| \geq 6)$.

## Problem 2 (30 points in total). A joint density on a triangle

The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 < x < 1, \quad 0 < y < 2(1-x), \\ 0, & \text{otherwise.} \end{cases}$$

**(a) (10 points)** Write the marginal PDF $f_X(x)$, including where it is zero.

**(b) (10 points)** Using the conditional PDF, calculate $P(Y > 1/2 \mid X = 1/2)$.

**(c) (10 points)** Calculate $P(X + Y/2 < 1/2)$.

## Problem 3 (30 points in total). MGFs and correlation of linear combinations

A random variable $R$ has PMF

$$p_R(r) = \begin{cases} 1/2, & r = -1, \\ 1/4, & r = 0, \\ 1/4, & r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $R_1, R_2$ be independent random variables with this same PMF, and define

$$S = 3 + R_1 - 2R_2.$$

**(a) (10 points)** Calculate $M_R(t)$ from the definition of an MGF, for all real $t$.

**(b) (10 points)** Calculate $M_S(t)$ for all real $t$, giving an explicit formula in terms of $t$. Show where independence is used.

**(c) (10 points)** Let $T = R_1 + R_2$. Calculate $\text{Corr}(T, S)$. Show your reasoning.

## Problem 4 (30 points in total). Sessions with varying success rates

At the start of each session, a success probability $Q$ is selected with

$$P(Q = 1/4) = P(Q = 3/4) = \frac{1}{2}.$$

Conditional on $Q = q$, four trials are independent and each succeeds with probability $q$. Let $K$ be the number of successes in the session. Different sessions, including their selections of $Q$, are independent and follow this same model.

**(a) (10 points)** Express $\mathrm{Var}(K \mid Q = q)$ as a function of $q$, for $q \in \{1/4, 3/4\}$.

**(b) (10 points)** Calculate $\mathrm{Var}(K)$.

**(c) (10 points)** For $n$ sessions, let $K_i$ be the number of successes in session $i$, and define

$$\overline{K}_n = \frac{1}{n} \sum_{i=1}^{n} K_i.$$

Apply Chebyshev's inequality directly with deviation threshold $1/2$. Find the smallest positive integer $n$ for which this bound guarantees

$$P\big(|\overline{K}_n - \mathbb{E}[K]| < 1/2\big) \geq 0.95.$$

## Problem 5 (10 bonus points in total). Optional bonuses

Complete the required problems first. A brief, well-supported argument is sufficient.

### (a) (5 bonus points). A joint tail bound

Let $X, Y$ be nonnegative random variables with finite second moments such that

$$\mathbb{E}[X] = \mathbb{E}[Y] = 1, \qquad \mathrm{Cov}(X, Y) = 0.$$

Independence is not assumed.

(i) Find the largest possible value of $P(X \geq 2,\ Y \geq 2)$.
(ii) Give a joint PMF attaining this value and verify the stated conditions.
(iii) Prove that no larger value is possible.

### (b) (5 bonus points). Constant conditional means

Construct two nonconstant random variables $X, Y$, supported on finitely many pairs, such that $\mathbb{E}[Y \mid X = x] = 0$ for every $x$ with $P(X = x) > 0$, but $X, Y$ are not independent.

(i) Give a joint PMF for $X, Y$.
(ii) Verify the conditional-mean condition.
(iii) Demonstrate that $X, Y$ are dependent.