STA 130A: Mathematical Statistics: Brief Course

Practice midterm 2 – Set B

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Name: _____ **Student ID:** _____

Instructions: This practice midterm is **closed-book**, except for the permitted note sheet described below. You may bring a pen or pencil, one letter-sized sheet of *hand-written* notes (both sides), and a *non-graphing* calculator. No other materials (e.g., textbooks) are allowed. For a timed attempt, allow 50 minutes. The **total score is 120 points**, with *up to 10 bonus points available.* Please confirm that you have all 6 pages.

- Make sure to clearly write your name and student ID above.
- Present answers succinctly, but include all relevant steps for full credit. Partial credit is possible only if your reasoning is clearly shown and traceable by the grader.
- If necessary, round numerical answers to three decimal places. Exact expressions, including fractions and factored expressions, are acceptable.
- Bonus problems can be more challenging; consider attempting them after you finish the main problems.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5 (bonus)	
Total	

Problem 1 (30 points in total). Conceptual checks

Treat each item separately. One or two lines of calculation or justification suffice.

(a) (5 points) A joint PDF $f_{X,Y}$ is positive whenever $x > 0$, $y > 0$, and $x + y < 1$, and is zero elsewhere. Can X, Y be independent? Explain in one sentence.

(b) (5 points) Suppose $W \geq 0$, $\mathbb{E}[W] = 2$, and $\text{Var}(W) = 1$. Which gives the smaller upper bound for $P(W \geq 6)$: Markov's inequality applied directly to W, or Chebyshev's inequality? Report **the inequality and its numerical bound**.

(c) (5 points) Suppose $\text{Var}(X) = \text{Var}(Y) = 1$ and $\text{Var}(X - Y) = 1/2$. Calculate $\text{Corr}(X, Y)$.

(d) (5 points) Let X, Y have finite variances, with $\text{Var}(X) = 1$ and $\mathbb{E}[Y \mid X] = 2X$. Could $\text{Var}(Y) = 3$? Justify your answer in one line.

(e) (5 points) Let X have an MGF finite near 0. A fair coin, tossed independently of X, sets $Z = X$ on heads and $Z = -X$ on tails. Express $M_Z(t)$ in terms of M_X, for t sufficiently close to 0.

(f) (5 points) The MGFs of X and Y are finite and equal on an open interval containing 0. Must $P(X = Y) = 1$? Explain in one sentence.

Problem 2 (30 points in total). Continuous distributions

The three parts use **separate probability models**. For every PDF requested below, specify where it is nonzero and give its value elsewhere.

(a) (10 points) The joint PDF of X, Y is

$$f_{X,Y}(x,y) = \begin{cases} x+y, & 0<x<1, \quad 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

Write the marginal PDF $f_X(x)$.

(b) (10 points) A different pair U, V has joint PDF

$$f_{U,V}(u,v) = \begin{cases} 6u, & 0<u<v<1, \\ 0, & \text{otherwise.} \end{cases}$$

For $0<v<1$, write the conditional PDF $f_{U|V}(u \mid v)$ as a function of u.

(c) (10 points) Separately, let $W \sim \text{Uniform}(-2,2)$ and $Z = W^2$. Write the PDF $f_Z(z)$.

Problem 3 (30 points in total). MGFs and covariance of linear combinations

For this problem, you may use the following formulas, valid for all real t:

$$N \sim N(\mu, \sigma^2) \quad \Longrightarrow \quad M_N(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right),$$
$$J \sim \text{Poisson}(\lambda) \quad \Longrightarrow \quad M_J(t) = \exp\big(\lambda(e^t - 1)\big).$$

In $N(\mu, \sigma^2)$, the second parameter is the variance.

For parts (a)–(b): Let $N_1 \sim N(1, 4)$ and $N_2 \sim N(-1, 1)$ be independent, and define $H = 2N_1 - N_2$.

(a) (10 points) Use MGFs to identify the distribution of H by its name and parameters. Show your MGF calculation and where independence is used.

(b) (10 points) Calculate $\text{Cov}(N_1 + N_2, H)$. Show your reasoning.

(c) (10 points) Separately, let $A \sim \text{Poisson}(2)$ and $B \sim \text{Poisson}(3)$ be independent. Use MGFs to identify the distribution of $A + B$ by its name and parameters. Show your MGF calculation.

Problem 4 (30 points in total). A shared calibration error

Measurements of a fixed quantity θ satisfy

$$W_i = \theta + B + \varepsilon_i, \qquad i = 1, 2, \ldots.$$

The same random offset B is shared by all measurements. Assume that $B, \varepsilon_1, \varepsilon_2, \ldots$ are mutually independent, with

$$\mathbb{E}[B] = 0, \quad \mathrm{Var}(B) = 4, \qquad \mathbb{E}[\varepsilon_i] = 0, \quad \mathrm{Var}(\varepsilon_i) = 1.$$

For a positive integer n, define $\overline{W}_n = \frac{1}{n}\sum_{i=1}^{n} W_i$. No distributional forms are assumed beyond these conditions.

(a) (10 points) Calculate $\mathrm{Corr}(W_1, W_2)$. Show your reasoning.

(b) (10 points) Use the law of total variance to calculate $\mathrm{Var}(\overline{W}_n)$ as a function of n.

(c) (10 points)

(i) Use Chebyshev's inequality to obtain a lower bound, as a function of n, for

$$P\big(|\overline{W}_n - \theta| < 4\big).$$

(ii) Does this bound guarantee a probability of at least 0.95 for any positive integer n? Explain in one sentence.

Problem 5 (10 bonus points in total). Optional bonuses

Complete the required problems first. A brief, well-supported argument is sufficient.

(a) (5 bonus points). Three pairwise correlations

Each of X_1, X_2, X_3 takes the values -1 and 1 with probability $1/2$. Suppose

$$\mathrm{Corr}(X_1, X_2) = \mathrm{Corr}(X_1, X_3) = \mathrm{Corr}(X_2, X_3) = \rho.$$

(i) Find the smallest possible value of ρ.
(ii) Give a joint PMF attaining it and verify the marginal distributions and correlations.
(iii) Prove that no smaller value is possible.

(b) (5 bonus points). Two conditional-mean identities

Let X, Y have finite second moments and satisfy

$$\mathbb{E}[Y \mid X] = X, \qquad \mathbb{E}[X \mid Y] = Y.$$

Must $P(X = Y) = 1$? Prove your answer or give a counterexample.