

# STA 130A – Homework 4

Submission due: Thu, October 29 at 11:59 PM PT

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**Instructions:** Upload a single PDF file to Gradescope via Canvas (“Homework 4” under “Assignments”). Name the file using the prefix of your UC Davis email ID and the homework number (e.g., `dgsong_hw4.pdf`). Include “STA 130A,” your name, and the last four digits of your student ID on the front page. No late submissions will be accepted; submissions after the deadline receive 0 points. For full submission requirements and the late submission policy, see the syllabus.

## Problem 1 (30 points). Joint densities, probabilities, and expectations

Suppose that  $X$  and  $Y$  have joint PDF as follows: for some  $c > 0$ ,

$$f_{X,Y}(x,y) = \begin{cases} c(x+2y), & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) **(10 points)** Find  $c$  and the marginal PDFs  $f_X$  and  $f_Y$ . Sketch the part of the unit square where  $x + y \leq 1$ , and compute  $P(X + Y \leq 1)$ . What is  $P(X + Y = 1)$ ? Briefly explain.
- (b) **(10 points)** Find the joint CDF  $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$  for  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ . Determine whether  $X$  and  $Y$  are independent, and justify your answer.
- (c) **(10 points)** Compute

$$\mathbb{E}[X^2 + Y], \quad \mathbb{E}[XY], \quad \mathbb{E}[\max\{X, Y\}].$$

For the first expectation, you may use the marginal PDFs and linearity. For the other two, integrate directly against the joint PDF.

## Problem 2 (30 points). Conditioning on a value or an event

Suppose that  $(X, Y)$  is uniformly distributed over

$$T = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 3x\}.$$

Thus  $f_{X,Y}(x,y) = c$  on  $T$  and is zero elsewhere, for some  $c > 0$ .

- (a) **(10 points)** Sketch  $T$ . Find  $c$  and the marginal PDFs  $f_X$  and  $f_Y$ .
- (b) **(10 points)** For  $0 < x < 1$ , find  $f_{Y|X}(y | x)$ , including its support, and  $\mathbb{E}[Y | X = x]$ . Then use the law of total expectation to compute  $\mathbb{E}[Y]$ , and obtain  $\mathbb{E}[X + 2Y]$ .
- (c) **(10 points)** Let  $A = \{Y > 3/2\}$ . Find  $f_{X|A}(x)$  and  $f_{X|Y}(x | 3/2)$ , including their supports. Explain geometrically why learning  $Y > 3/2$  affects the distribution of  $X$  differently from learning  $Y = 3/2$ .  
*Hint:* For  $A$ , integrate the joint density over  $y > 3/2$ .

**Problem 3 (20 points). Functions of one random variable**

- (a) **(10 points)** Adapted from [BT08, Chapter 4, Problem 2, p. 246]. Let  $U \sim \text{Uniform}(0, 1)$  and  $V = e^U$ . Find the CDF of  $V$ , then differentiate to obtain its PDF.
- (b) **(10 points)** Let  $X \sim \text{Uniform}(-1, 2)$  and  $Z = X^2$ . Find the CDF and PDF of  $Z$ . Explain why the PDF changes form within its support.
- Hint:* When rewriting  $\{X^2 \leq z\}$ , account for the support  $[-1, 2]$  of  $X$ .

**Problem 4 (20 points). Sums and minima**

- (a) **(10 points)** Let  $U$  and  $V$  be independent  $\text{Uniform}(0, 1)$  random variables, and let  $S = U + V$ . Find the CDF and PDF of  $S$ .
- Hint:* Sketch  $\{(u, v) : u + v \leq s\}$  within the unit square. Consider  $0 < s \leq 1$  and  $1 < s < 2$  separately.
- (b) **(10 points)** Two components have independent lifetimes  $X \sim \text{Exponential}(\lambda)$  and  $Y \sim \text{Exponential}(\mu)$ , where  $\lambda, \mu > 0$  are rates. The time of the first failure is  $T = \min\{X, Y\}$ . Find the CDF and PDF of  $T$ , and the probability  $P(X < Y)$  that the first component fails first.
- Hint:* For  $t \geq 0$ , use  $P(T > t) = P(X > t, Y > t)$ .

**Problem 5\* (up to 10 bonus points).**

Each part is optional and worth up to 5 additional points.

- (a) **(5 bonus points)** Let  $U$  and  $V$  be independent  $\text{Uniform}(0, 1)$  random variables, and let  $D = |U - V|$ . Find the CDF and PDF of  $D$ , and compute  $\mathbb{E}[D]$ .
- Hint:* Use areas in the unit square; the complement of  $\{D \leq d\}$  may be easier to describe.
- (b) **(5 bonus points)** Return to the triangular model in Problem 2. Define  $R = Y/(3X)$  for  $X > 0$  and set  $R = 0$  when  $X = 0$ , an event of probability zero. Find the CDF and PDF of  $R$ . Are  $R$  and  $X$  independent? Justify your answer.
- Hint:* First find  $P(R \leq t \mid X = x)$  for  $0 < x < 1$ . A multivariable change of variables is not needed.

**References**

- [BT08] Dimitri P. Bertsekas and John N. Tsitsiklis. *Introduction to Probability*, 2nd edition. Athena Scientific, 2008.