

STA 130A: Mathematical Statistics: Brief Course

Lecture 9: Continuous Random Variables, PDFs, and CDFs

Dogyoon Song

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Announcements

Midterm 1 is in class on Mon, Oct 19

- You may bring *one **hand-written** letter-sized (8.5×11 inches), double-sided sheet of paper* with formulas and brief notes
- **Calculator:** Simple (non-graphing) calculators only
- **No textbooks** or other study aids beyond the permitted note sheet and calculator
- **SDC accommodations:** Confirm scheduling with AES online

Agenda

Last time:

- Conditional expectation and the law of total expectation
- Independence of random variables

Today:

- Continuous random variables and probability density functions
- Expectation and variance
- Cumulative distribution functions

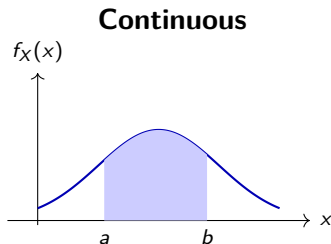
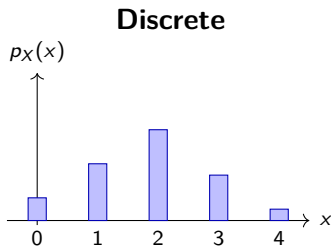
Continuous random variables: From sums to areas

For a discrete random variable, probabilities are given as a sum of point masses:

$$P(X \in A) = \sum_{x \in A} p_X(x).$$

For a continuous random variable, probabilities are given by **areas under a density curve**:

$$P(X \in A) = \int_A f_X(x) dx.$$



Continuous random variable and PDF

Definition (Continuous random variable)

A random variable X is **continuous** if there exists a nonnegative function f_X , called the **probability density function** or **PDF**, such that^a

$$P(X \in B) = \int_B f_X(x) dx \quad \text{for every interval } B \subseteq \mathbb{R}.$$

^aA rigorous formulation requires measure theory; for this course, you may think of B as an interval in $\mathbb{R}$.

A valid PDF satisfies

$$f_X(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f_X(x) dx = 1.$$

For any $a, b \in \mathbb{R}$ with $a \leq b$,

$$P(X = a) = \int_a^a f_X(x) dx = 0 \quad \text{and} \quad P(a \leq X \leq b) = \int_a^b f_X(x) dx.$$

Important: The density value $f_X(x)$ may exceed 1; probabilities are given by areas, not heights

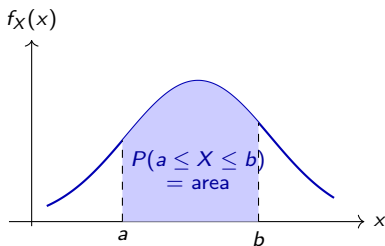
Visual interpretation of a PDF

A PDF assigns probability through **area**:

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx.$$

For a very small interval of width $\Delta x > 0$,

$$P(x \leq X \leq x + \Delta x) \approx f_X(x) \Delta x.$$



Density measures probability per unit length; a larger density means more probability mass *nearby*, but $f_X(x)$ itself is not a probability

Consequence: For a continuous random variable,

$$P(X = x) = 0$$

for every single point x , even though intervals can have positive probability. Therefore,

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b),$$

i.e., for continuous random variables, endpoint conventions do not matter.

Example: Uniform random variable

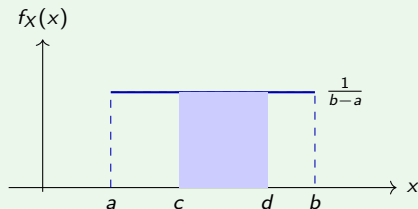
Example (Uniform random variable on an interval)

For $a < b$, X is **uniform** on $[a, b]$, written

$$X \sim \text{Uniform}(a, b),$$

if its PDF is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$



For any subinterval $[c, d] \subseteq [a, b]$,

$$P(c \leq X \leq d) = \int_c^d \frac{1}{b-a} dx = \frac{d-c}{b-a}.$$

So under a uniform model, probability is proportional to interval length.

Example: Piecewise-constant PDF (1/2)

Example (Piecewise constant PDF)

Let

$$f_X(x) = \begin{cases} c, & 0 \leq x < 1, \\ 2c, & 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Question: Find the constant c , and compute $P(0.5 \leq X \leq 1.5)$.

1) Because the total area must be 1,

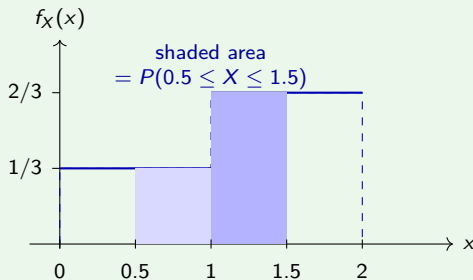
$$\int_{-\infty}^{\infty} f_X(x) dx = c(1 - 0) + 2c(2 - 1) = 3c = 1, \quad \implies \quad c = \frac{1}{3}.$$

2) Then

$$P(0.5 \leq X \leq 1.5) = \int_{0.5}^{1.5} f_X(x) dx = \int_{0.5}^1 \frac{1}{3} dx + \int_1^{1.5} \frac{2}{3} dx = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

Example: Piecewise-constant PDF (2/2)

Example (Piecewise constant PDF, continued)



$$P(0.5 \leq X \leq 1.5) = \int_{0.5}^{1.5} f_X(x) dx = \int_{0.5}^1 \frac{1}{3} dx + \int_1^{1.5} \frac{2}{3} dx = \underbrace{0.5 \cdot \frac{1}{3}}_{1/6} + \underbrace{0.5 \cdot \frac{2}{3}}_{1/3} = \frac{1}{2}.$$

Expectation and variance: Continuous case

For a continuous random variable with PDF f_X , the **expectation** of X is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

For any function $g : \mathbb{R} \rightarrow \mathbb{R}$,

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

The **variance** of X is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

Discrete vs continuous analogy

$$\sum_x \longleftrightarrow \int dx, \quad p_X(x) \longleftrightarrow f_X(x).$$

Example: Mean and variance of a uniform random variable

Example (Mean and variance of a uniform random variable)

Let $X \sim \text{Uniform}(a, b)$. Then

$$f_X(x) = \frac{1}{b-a}, \quad a \leq x \leq b.$$

The mean is

$$\mathbb{E}[X] = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{x^2}{2} \Big|_a^b = \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} = \frac{a+b}{2}.$$

Also,

$$\mathbb{E}[X^2] = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{x^3}{3} \Big|_a^b = \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3} = \frac{a^2 + ab + b^2}{3}.$$

Therefore,

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{(b-a)^2}{12}.$$

Example: Exponential random variable

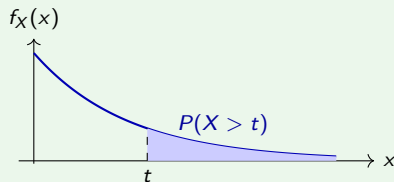
Example (Exponential random variable)

A random variable X has an **exponential distribution** with rate $\lambda > 0$, written

$$X \sim \text{Exponential}(\lambda),$$

if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$



First check normalization:

$$\int_0^{\infty} \lambda e^{-\lambda x} dx = [-e^{-\lambda x}]_0^{\infty} = 1.$$

For $t \geq 0$,

$$P(X > t) = \int_t^{\infty} \lambda e^{-\lambda x} dx = e^{-\lambda t}.$$

Note: Exponential RVs are commonly used to model waiting times until an occurrence. 12 / 28

Example: Mean of an exponential random variable

Example (Mean of an exponential random variable)

If $X \sim \text{Exponential}(\lambda)$, then

$$\mathbb{E}[X] = \int_0^{\infty} x \lambda e^{-\lambda x} dx.$$

Using integration by parts ($\int u \cdot v' dx = u \cdot v - \int u' \cdot v dx$),

$$\int_0^{\infty} x \lambda e^{-\lambda x} dx = -x e^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = 0 + \frac{1}{\lambda} = \frac{1}{\lambda}.$$

Thus,

$$\mathbb{E}[X] = \frac{1}{\lambda}.$$

Example: If events occur at average rate $\lambda = 2$ per hour, then the average waiting time is

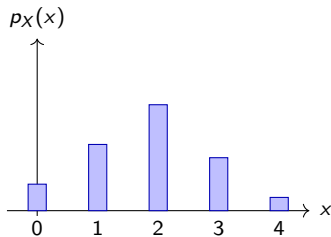
$$\frac{1}{2} \text{ hour} = 30 \text{ minutes}.$$

Motivation: Why another object?

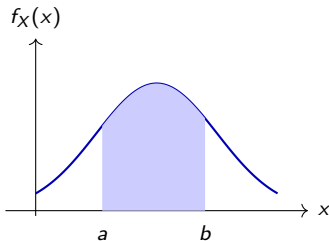
PMFs and PDFs look different:

$$P(X \in A) = \sum_{x \in A} p_X(x) \quad \text{versus} \quad P(X \in A) = \int_A f_X(x) dx.$$

Discrete



Continuous



Question: Is there one unified way to describe both types of random variables?

Cumulative distribution function

Definition (Cumulative distribution function)

For any random variable X , the **cumulative distribution function** or **CDF** is

$$F_X(x) = P(X \leq x), \quad x \in \mathbb{R}.$$

Interpretation:

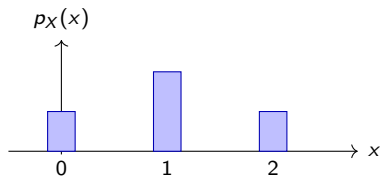
- $F_X(x)$ accumulates probability up to the point x .
- Its value always corresponds to a probability, hence $0 \leq F_X(x) \leq 1$.
- It describes the distribution of X completely.

Useful identity: for $a \leq b$,

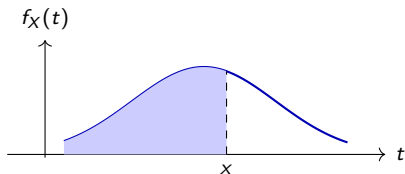
$$P(a < X \leq b) = F_X(b) - F_X(a).$$

Illustration: CDF as accumulated probability

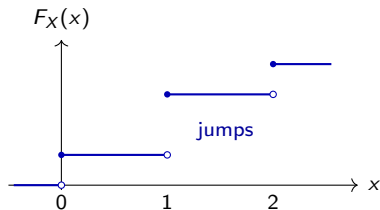
Discrete: PMF



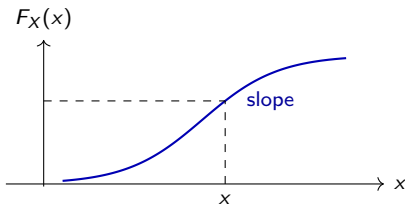
Continuous: PDF



Discrete: CDF



Continuous: CDF



Discrete mass $\leftrightarrow$ jumps in F_X .

Continuous density $\leftrightarrow$ slope of F_X .

CDF from a PMF: Discrete random variables

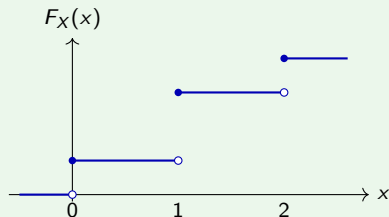
Example (Discrete random variable)

Let X have PMF

$$P(X = 0) = \frac{1}{4}, \quad P(X = 1) = \frac{1}{2}, \quad P(X = 2) = \frac{1}{4}.$$

Then

$$F_X(x) = P(X \leq x) = \sum_{x' \leq x} p_X(x')$$
$$= \begin{cases} 0, & x < 0, \\ \frac{1}{4}, & 0 \leq x < 1, \\ \frac{3}{4}, & 1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$



Interpretation:

- The CDF accumulates probability.
- The PMF is the jump in the CDF; large probability mass $\leftrightarrow$ larger jump.

CDF from a PDF: Continuous random variables

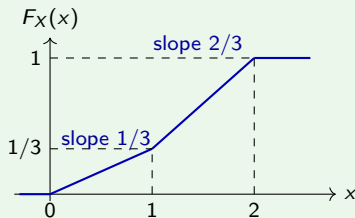
Example (Continuous random variable)

Let X have PDF

$$f_X(x) = \begin{cases} 1/3, & 0 \leq x < 1, \\ 2/3, & 1 \leq x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} F_X(x) &= P(X \leq x) = \int_{-\infty}^x f_X(t) dt \\ &= \begin{cases} 0, & x < 0, \\ \frac{1}{3}x, & 0 \leq x < 1, \\ \frac{2}{3}x - \frac{1}{3}, & 1 \leq x < 2, \\ 1, & x \geq 2. \end{cases} \end{aligned}$$



Interpretation:

- The PDF is the slope of the CDF; large density $\leftrightarrow$ rapidly increasing CDF.

Properties of CDFs

For any random variable X , its CDF $F_X(x) = P(X \leq x)$ satisfies:

- **Monotonicity:**

$$x \leq y \implies F_X(x) \leq F_X(y).$$

- **Limits:**

$$\lim_{x \rightarrow -\infty} F_X(x) = 0, \quad \lim_{x \rightarrow \infty} F_X(x) = 1.$$

- **Right-continuity:** $F_X(x) = \lim_{x' \rightarrow x+} F_X(x')$

- **Interval probabilities:**

$$P(a < X \leq b) = F_X(b) - F_X(a), \quad a \leq b.$$

- **How the CDF behaves:**

- Discrete RV: probability appears as jumps of size $p_X(x)$
- Continuous RV: continuous CDF, with $F'_X(x) = f_X(x)$ wherever f_X is continuous

Recovering PMFs and PDFs from CDFs

Discrete case: If X has possible values $x_1 < x_2 < \dots$, then

$$P(X = x_i) = F_X(x_i) - F_X(x_i^-),$$

where $F_X(x_i^-) := \lim_{x' \rightarrow x_i^-} F_X(x')$ denotes the value just before the jump.

Continuous case: If F_X is differentiable, then

$$f_X(x) = F'_X(x) = \frac{dF_X(x)}{dx}.$$

Upshot:

- Discrete probability mass appears as **jumps** in the CDF.
- Continuous density appears as the **slope** of the CDF.

Example: Deriving PDF from CDF (1/2)

Example (Maximum of two uniform points)

Let U_1, U_2 be independent $\text{Uniform}(0, 1)$, and let

$$M = \max\{U_1, U_2\}.$$

Question: Find the CDF and the PDF of M .

For $0 \leq m \leq 1$,

$$\begin{aligned} F_M(m) &= P(M \leq m) \\ &= P(U_1 \leq m, U_2 \leq m) \\ &= P(U_1 \leq m)P(U_2 \leq m) \\ &= m^2. \end{aligned}$$

Thus,

$$F_M(m) = \begin{cases} 0, & m < 0, \\ m^2, & 0 \leq m < 1, \\ 1, & m \geq 1. \end{cases}$$

Example: Deriving PDF from CDF (2/2)

Example (Maximum of two uniform points, continued)

Continuing:

$$F_M(m) = \begin{cases} 0, & m < 0, \\ m^2, & 0 \leq m < 1, \\ 1, & m \geq 1. \end{cases}$$

Since F_M is differentiable on $(0, 1)$,

$$f_M(m) = F'_M(m) = 2m, \quad 0 < m < 1.$$

Therefore,

$$f_M(m) = \begin{cases} 2m, & 0 \leq m < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Takeaway: It is sometimes easier to compute the CDF first and then differentiate.

Pop-up quiz 1: Probability from CDF

Let X have CDF

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x/2, & 0 \leq x < 1, \\ 3/4, & 1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

Which statement is correct?

- A) $P(X = 1) = 0$
- B) $P(X = 1) = 1/4$
- C) $P(0 < X < 1) = 1$
- D) F_X cannot be a CDF because it is flat on $[1, 2)$

Answer: B.

The jump at $x = 1$ is $F_X(1) - F_X(1^-) = 3/4 - 1/2 = 1/4$.

Follow-up: Is this random variable discrete, continuous, or neither? What feature of the CDF tells you that?

Pop-up quiz 2: Computing CDF

Let U_1, U_2 be independent $\text{Uniform}(0, 1)$, and let

$$N = \min\{U_1, U_2\}.$$

For $0 \leq n \leq 1$, which expression equals $F_N(n) = P(N \leq n)$?

- A) n^2
- B) $1 - (1 - n)^2$
- C) $1 - n^2$
- D) $(1 - n)^2$

Answer: B.

It is easier to use the complement:

$$P(N \leq n) = 1 - P(N > n) = 1 - P(U_1 > n, U_2 > n) = 1 - (1 - n)^2.$$

Follow-up: How would you obtain the PDF of N from this CDF?

Geometric and exponential CDFs

Geometric

Let $N \sim \text{Geometric}(p)$, where N is the trial number of the first success.

$$\begin{aligned}p_N(k) &= (1-p)^{k-1}p, \\P(N > n) &= \sum_{k=n+1}^{\infty} (1-p)^{k-1}p \\&= (1-p)^n, \quad n \in \mathbb{N}.\end{aligned}$$

So as a function on $\mathbb{R}$,

$$F_N(x) = \begin{cases} 0, & x < 1, \\ 1 - (1-p)^{\lfloor x \rfloor}, & x \geq 1. \end{cases}$$

Exponential

Let $X \sim \text{Exponential}(\lambda)$, with $\lambda > 0$, be an elapsed waiting time.

$$\begin{aligned}f_X(x) &= \lambda e^{-\lambda x}, \\P(X > x) &= \int_x^{\infty} \lambda e^{-\lambda x'} dx' \\&= e^{-\lambda x}, \quad x \geq 0.\end{aligned}$$

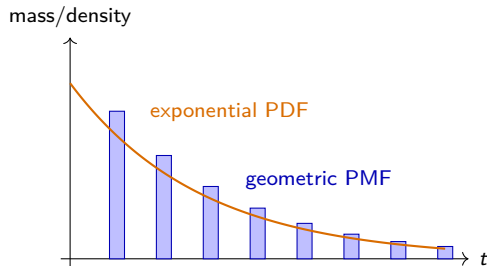
Thus,

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

Connection: Both random variables describe waiting times.

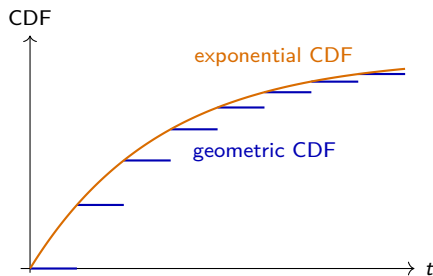
Illustration: Comparing geometric vs. exponential CDFs

PMF/PDF



Use $p = 0.3$ and $\lambda = -\log(1 - p)$.

CDF



Step CDF vs. smooth CDF

With $\lambda = -\log(1 - p)$, the two survival probabilities agree at integer times:

$$e^{-\lambda n} = (1 - p)^n.$$

Wrap-up

PDFs: probabilities are areas, not heights

- $f_X(x) \geq 0$, $\int_{-\infty}^{\infty} f_X(x) dx = 1$, and $P(X = x) = 0$
- $P(a \leq X \leq b) = \int_a^b f_X(x) dx$; endpoints do not affect this probability

Expectations and variances: replace sums by integrals

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx, \quad \text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

CDFs: one description for every distribution

- $F_X(x) = P(X \leq x)$; $P(a < X \leq b) = F_X(b) - F_X(a)$ for $a \leq b$
- Point masses are jumps; for a PDF, $F_X(x) = \int_{-\infty}^x f_X(t) dt$

Suggested reading: [BT08, Ch. 3.1–3.2]

Next lecture: Normal distributions, standardization, probabilities, and quantiles

References



Dimitri Bertsekas and John N Tsitsiklis.

Introduction to probability, volume 1.

Athena Scientific, 2nd edition, 2008.