

STA 130A: Mathematical Statistics: Brief Course

Lecture 10: Normal Random Variables

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Announcements

Homework 3 is due **Thu, Oct 15, 11:59 PM PT**

- Please submit on time and follow the submission instructions
 - To Gradescope via Canvas in a single PDF
 - Email submissions are not accepted

Lectures on **Wed, Oct 21** and **Fri, Oct 23** will be held via Zoom; details and the Zoom link will be posted on Canvas

Agenda

Last time: PDFs and CDFs

- $F_X(x) = P(X \leq x)$
- $P(a < X \leq b) = F_X(b) - F_X(a)$
- PMF values are CDF jumps; a continuous PDF is the CDF's derivative

Today: Normal random variables

- Normal PDF and its parameters
- Standard normal distribution and the normal table
- Standardization, probabilities, and quantiles

Recap: Cumulative distribution functions

For any random variable X ,

$$F_X(x) = P(X \leq x).$$

- $P(a < X \leq b) = F_X(b) - F_X(a)$

For a discrete random variable,

$$F_X(x) = \sum_{x' \leq x} p_X(x'), \quad p_X(x_i) = F_X(x_i) - F_X(x_i^-),$$

where $F_X(x_i^-) := \lim_{x' \rightarrow x_i^-} F_X(x')$ is the value just before the jump.

For a continuous random variable with PDF f_X ,

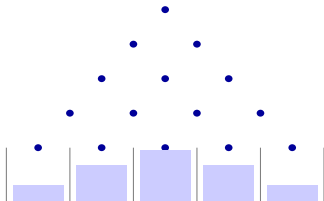
$$F_X(x) = \int_{-\infty}^x f_X(t) dt, \quad f_X(x) = F'_X(x)$$

when F_X is differentiable.

Motivation: Where do normal random variables arise?

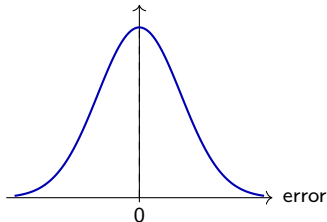
Many real-world quantities are approximately bell-shaped.

Galton board



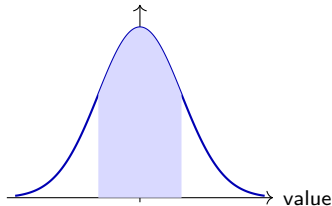
With many independent fair deflections, the binomial landing distribution is approximately normal.

Measurement error



Small positive and negative errors often cluster around 0.

Natural variation



Heights, test-score noise, and many biological traits are often approximately normal.

Common pattern: values near the center are most common, while very small or very large values are less common.

Motivation: Why does the bell shape appear so often?

A common mechanism is the accumulation of **many small influences added together**.

Example (Examples of many-small-effects behavior)

- **Galton board:** each bounce sends the ball slightly left or right.
- **Measurement error:** many tiny sources of error combine.
- **Human height:** many genetic and environmental factors contribute.

$$X = X_1 + X_2 + \cdots + X_n$$

- This is one reason the normal distribution is so important in probability and statistics
- Later in the course, this idea connects to the **central limit theorem**
- Normal approximation requires suitable assumptions though; many summands alone are not enough

For now, we focus on [how to describe and work with the normal model itself](#).

Normal random variable

Definition (Normal random variable)

A random variable X is **normal** with mean μ and variance σ^2 , written

$$X \sim N(\mu, \sigma^2),$$

if its PDF is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}, \quad x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

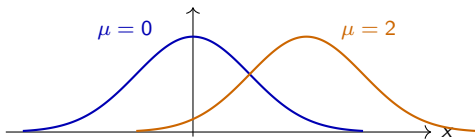
- $\mathbb{E}[X] = \mu$: location/center
- $\text{Var}(X) = \sigma^2$, so σ is the standard deviation/spread parameter

Normal PDF: Shape and parameters

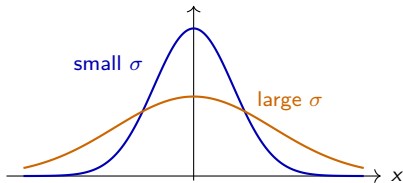
Properties of the normal PDF

- Symmetric around μ
- Bell-shaped
- Total area under the curve is 1
- μ controls location; σ controls spread

Changing μ : shift



Changing σ : spread



Normality preserved under affine transformations

Normality is preserved under affine transformations

For any a, b with $a \neq 0$, if $X \sim N(\mu, \sigma^2)$, then

$$aX + b \sim N(a\mu + b, a^2\sigma^2).$$

It is not surprising from the properties of expectation/variance that

$$\mathbb{E}[aX + b] = a\mu + b, \quad \text{Var}(aX + b) = a^2\sigma^2.$$

Why is this true? One can verify it by deriving the PDF of $aX + b$.

- We will develop the general transformation method later

Implication: every normal random variable can be converted into a **standard** normal random variable.

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

Standard normal random variable

Definition (Standard normal)

The **standard normal** random variable is

$$Z \sim N(0, 1).$$

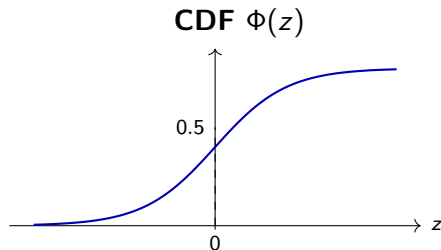
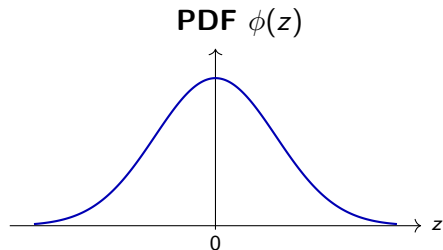
Its PDF and CDF are respectively denoted by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad \Phi(z) = P(Z \leq z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

There is no simple elementary formula for $\Phi(z)$

- We can compute it from the PDF using the defining integral.
- Typically we use a standard normal table, calculator, or software.

Standard normal PDF and CDF: Visual summary



By symmetry,

$$\Phi(-z) = 1 - \Phi(z).$$

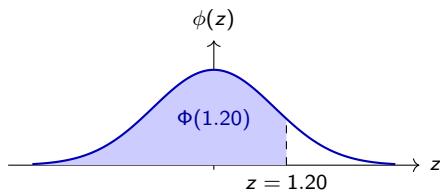
Useful consequences:

$$\Phi(0) = \frac{1}{2}, \quad P(Z > z) = 1 - \Phi(z), \quad P(|Z| \leq z) = 2\Phi(z) - 1 \quad (z \geq 0).$$

Standard normal table

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574

What the table gives



Example table entry

$$\Phi(1.20) = P(Z \leq 1.20) \approx 0.8849.$$

z	1.18	1.19	1.20
$\Phi(z)$	0.8810	0.8830	0.8849

Standardization

Definition (Standardization)

If $X \sim N(\mu, \sigma^2)$, then we can “standardize” it by defining

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

Observe that for any $x \in \mathbb{R}$,

$$X \leq x \quad \Longleftrightarrow \quad Z \leq \frac{x - \mu}{\sigma}$$

Therefore,

$$\begin{aligned} P(a < X \leq b) &= P\left(\frac{a - \mu}{\sigma} < \frac{X - \mu}{\sigma} \leq \frac{b - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right). \end{aligned}$$

One normal table is enough for all normal distributions.

Using the standard normal table

Typical workflow:

1. Convert X to $Z = (X - \mu)/\sigma$.
2. Rewrite the probability in terms of Z .
3. Use the table for Φ .

Commonly used identities:

$$P(Z \leq z) = \Phi(z),$$

$$P(Z > z) = 1 - \Phi(z),$$

$$P(a < Z \leq b) = \Phi(b) - \Phi(a).$$

$$\because \Phi(-z) = 1 - \Phi(z)$$

Remark: Most normal tables give left-tail probabilities $\Phi(z) = P(Z \leq z)$.

- For right tails or middle intervals, first rewrite the probability in terms of Φ .

Example: Normal-table calculation

Example

Question: Suppose

$$X \sim N(70, 10^2).$$

Compute $P(60 \leq X \leq 85)$ (for a continuous random variable, endpoints do not matter).

Standardize X :

$$P(60 \leq X \leq 85) = P\left(\frac{60 - 70}{10} \leq Z \leq \frac{85 - 70}{10}\right) = P(-1 \leq Z \leq 1.5).$$

Thus,

$$P(-1 \leq Z \leq 1.5) = \Phi(1.5) - \Phi(-1).$$

Using $\Phi(1.5) \approx 0.9332$ and $\Phi(-1) \approx 0.1587$,

$$P(60 \leq X \leq 85) \approx 0.7745.$$

Example: Tail probability

Example

Question: Let

$$X \sim N(100, 15^2).$$

Compute $P(X > 130)$.

$$P(X > 130) = P\left(Z > \frac{130 - 100}{15}\right) = P(Z > 2).$$

Thus,

$$P(Z > 2) = 1 - \Phi(2).$$

Using $\Phi(2) \approx 0.9772$,

$$P(X > 130) \approx 0.0228.$$

Interpretation: A value 2 standard deviations above the mean is relatively rare.

Normal quantiles: Finding a cutoff

For $X \sim N(\mu, \sigma^2)$ and $0 < q < 1$, the **q -quantile** x_q satisfies

$$P(X \leq x_q) = q.$$

Standardization gives

$$z_q = \Phi^{-1}(q), \quad x_q = \mu + \sigma z_q.$$

Locate the probability q inside the normal table; interpolate if needed to find z_q .

Example (95th percentile)

For $X \sim N(100, 15^2)$, use $z_{0.95} \approx 1.645$:

$$x_{0.95} \approx 100 + 15 \times 1.645 \approx 124.7.$$

About 95% of values lie below this cutoff, and 5% above.

Central 95% interval: Use $z_{0.975} \approx 1.96$, giving approximately $\mu \pm 1.96\sigma$ (2.5% in each tail).

Pop-up quiz 1: Computing normal probabilities

Suppose $X \sim N(10, 4)$.

Question: Which expression equals $P(X \leq 13)$?

- A) $\Phi(13)$
- B) $\Phi(1.5)$
- C) $\Phi(0.75)$
- D) $1 - \Phi(1.5)$

Answer: B.

Since the standard deviation is 2, we standardize and obtain

$$\frac{13 - 10}{2} = 1.5,$$

so

$$P(X \leq 13) = P(Z \leq 1.5) = \Phi(1.5).$$

Follow-up: What expression gives $P(X > 13)$?

Pop-up quiz 2: Computing normal probabilities

Let $Z \sim N(0, 1)$. Suppose $\Phi(1.2) = 0.8849$.

Question: Which expression equals $P(-1.2 \leq Z \leq 1.2)$?

- A) 0.8849
- B) $1 - 0.8849$
- C) $2(0.8849) - 1$
- D) $1 - 2(0.8849)$

Answer: C.

By symmetry, $\Phi(-1.2) = 1 - \Phi(1.2)$. Hence

$$P(-1.2 \leq Z \leq 1.2) = \Phi(1.2) - \Phi(-1.2) = 2\Phi(1.2) - 1.$$

Follow-up: Express $P(Z > 1.2)$ using $\Phi(1.2)$.

Additional exercise: Manufacturing tolerance (1/2)

Example (A more involved normal calculation)

A machine produces rods whose lengths are approximately normal:

$$X \sim N(10, 0.2^2)$$

in centimeters.

Suppose that a rod is considered acceptable if

$$9.7 \leq X \leq 10.3.$$

Questions:

1. What fraction of rods are acceptable?
2. Among 100 independently produced rods, what is the expected number acceptable?
3. With the mean kept at 10, if the standard deviation is increased, would this fraction increase or decrease?

Additional exercise: Manufacturing tolerance (2/2)

Example

Answer to Q1: Standardize X :

$$P(9.7 \leq X \leq 10.3) = P\left(\frac{9.7 - 10}{0.2} \leq Z \leq \frac{10.3 - 10}{0.2}\right) = P(-1.5 \leq Z \leq 1.5) = \Phi(1.5) - \Phi(-1.5).$$

Using symmetry, $\Phi(-1.5) = 1 - \Phi(1.5)$, and therefore,

$$P(-1.5 \leq Z \leq 1.5) = 2\Phi(1.5) - 1 \approx 2 \times 0.933 - 1 = 0.866.$$

Answer to Q2: If N is the number acceptable among 100 independently produced rods, then

$$N \sim \text{Binomial}(100, 0.866),$$

where 0.866 is the acceptance probability for one rod. Hence $\mathbb{E}[N] = 100 \times 0.866 = 86.6$.

Answer to Q3: With $\mu = 10$ fixed, the acceptance probability $2\Phi(0.3/\sigma) - 1$ decreases as σ increases. Hence the fraction also decreases.

Wrap-up

Normal random variable: $X \sim N(\mu, \sigma^2)$ has mean μ and variance σ^2

- The normal PDF is symmetric and bell-shaped.
- μ controls location; σ controls spread.

Standardization: If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1), \quad \text{and} \quad P(a < X \leq b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right).$$

What you should practice:

- Translate every normal probability into a standard normal probability.
- Use complements and symmetry to convert table values into the desired probability
- Be careful about variance vs. standard deviation and left-tail vs. right-tail probabilities.

Suggested reading: [BT08, Ch. 3.3]

Next lecture: Midterm 1 review

References



Dimitri Bertsekas and John N Tsitsiklis.

Introduction to probability, volume 1.

Athena Scientific, 2nd edition, 2008.