

STA 130A: Mathematical Statistics: Brief Course

Lecture 11: Review for Midterm 1

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Announcements

Midterm 1 is in class on Mon, Oct 19

- You may bring *one **hand-written** letter-sized (8.5×11 inches), double-sided sheet of paper* with formulas and brief notes
- **Calculator:** Simple (non-graphing) calculators only
- **No textbooks** or other study aids beyond the permitted note sheet and calculator
- **SDC accommodations:** Confirm scheduling with AES online
- Check the course webpage for practice materials

Lectures on **Wed, Oct 21** and **Fri, Oct 23** will be held via Zoom; see Canvas

Review for Midterm 1

What you should know cold:

- Translate a word problem into a sample space, events, and random variables.
- Identify the right tool: counting, conditioning, independence, PMFs/PDFs/CDFs, moments, and normal standardization.
- State the relevant formula with its assumptions before using it.
- Execute calculations cleanly and justify the key modeling step.

Today's review focuses on:

- Core definitions and formulas from Lectures 1–10.
- Typical problem types from lecture, homework, and the practice midterm.
- Common pitfalls that cause otherwise-correct solutions to fail.

A problem-solving recipe

Before computing, ask yourself:

1. **What is random?** Identify the experiment, outcomes, events, and random variables.
2. **What is given?** Probabilities, PMFs/PDFs/CDFs, parameters, independence, or equal likelihood?
3. **What is being asked?** A probability, distribution, mean, variance, conditional quantity?
4. **Which tool fits?** Counting, conditioning/Bayes, independence, expectation rules?
5. **What assumption justifies the step?** Support? Equal likelihood? Independence? Disjointness? Partition?

Good exam solutions are not just computations: They should clearly identify the event/model/tool being used. Fluency would matter, rather than mere memorization.

Review: Events and basic probability identities

Set theory:

- Events are sets: use unions, intersections, complements, and partitions carefully.
- Know these identities:

$$\begin{aligned}P(A^c) &= 1 - P(A), & P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\(A \cup B)^c &= A^c \cap B^c, & (A \cap B)^c &= A^c \cup B^c\end{aligned}$$

- Venn diagrams are often useful, but don't forget to justify using probability axioms

Probability axioms:

- Nonnegativity, additivity, normalization
- Probability is defined for events

Typical task: Rewrite a complicated event into simpler or disjoint pieces before computing probabilities; use complements or notice inclusion/inequality relations

Review: Conditional probability

- Definition:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad (P(B) > 0)$$

- Multiplicative rule:

$$P(A \cap B) = P(B) P(A \mid B) = P(A) P(B \mid A)$$

- Law of total probability: if $\{A_i\}$ is a partition with $P(A_i) > 0$, then

$$P(B) = \sum_i P(B \mid A_i) P(A_i)$$

- Bayes' rule: if $\{A_i\}$ is a partition and $P(B) > 0$, then

$$P(A_j \mid B) = \frac{P(B \mid A_j) P(A_j)}{\sum_i P(B \mid A_i) P(A_i)}$$

Typical task: Identify the observed event, condition on a useful partition, and weight conditional probabilities by the partition probabilities.

Review: Independence of events

- Definition:

$$P(A \cap B) = P(A) P(B)$$

- If $P(B) > 0$, this is equivalent to

$$P(A \mid B) = P(A)$$

- Independence is different from disjointness: if $A \cap B = \emptyset$ and $P(A), P(B) > 0$, then A and B are *not* independent.
- For several events: pairwise independence does *not* imply mutual independence.
- Conditional independence given C , with $P(C) > 0$:

$$P(A \cap B \mid C) = P(A \mid C)P(B \mid C).$$

Conditioning can change independence.

Typical task: Test independence by comparing $P(A \cap B)$ with $P(A)P(B)$, or compare $P(A \mid B)$ with $P(A)$ when $P(B) > 0$.

Review: Counting toolkit

- Product rule: break the process into stages and multiply the number of choices.
- Permutations (order matters; no replacement; $0 \leq k \leq n$):

$$\frac{n!}{(n-k)!}$$

- Combinations (order does not matter):

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- For a finite sample space with equally likely outcomes:

$$P(A) = \frac{|A|}{|\Omega|}.$$

Typical task: Decide whether order matters, whether repetitions are allowed, and whether it is easier to count the complement.

Review: Discrete random variables and PMFs

- A random variable is a function from outcomes to numbers:

$$X : \Omega \rightarrow \mathbb{R}.$$

- A PMF is

$$p_X(x) = P(X = x),$$

with $p_X(x) \geq 0$ and $\sum_x p_X(x) = 1$.

- For a function $Y = g(X)$,

$$p_Y(y) = \sum_{x: g(x)=y} p_X(x).$$

- To compute a PMF, identify the event $\{X = x\}$ and add the probabilities of all outcomes therein.

Typical task: Derive the PMF of X or $g(X)$ from an experiment, and check that the PMF sums to 1.

Review: Expectation and variance

- Expectation:

$$\mathbb{E}[X] = \sum_x x p_X(x), \quad \mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$$

- Variance:

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

- Expectation and variance of an affine function:

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b, \quad \text{Var}(aX + b) = a^2 \text{Var}(X)$$

- Expectation is linear even without independence.
- Caution: in general, $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$.
- Indicator trick: if $X = \sum_i I_i$, then $\mathbb{E}[X] = \sum_i \mathbb{E}[I_i] = \sum_i P(I_i = 1)$.

Typical task: Compute $\mathbb{E}[X]$, $\mathbb{E}[g(X)]$, and $\text{Var}(X)$; use indicators or transformations when direct computation is easier.

Review: Some named discrete models

- **Bernoulli**(p): one success/failure trial

$$P(X = 1) = p, \quad P(X = 0) = 1 - p, \quad \mathbb{E}[X] = p, \quad \text{Var}(X) = p(1 - p)$$

- **Binomial**(n, p): successes in n independent trials with common success probability p

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \dots, n, \quad \mathbb{E}[X] = np, \quad \text{Var}(X) = np(1-p)$$

- **Geometric**(p): trials through the first success in independent Bernoulli(p) trials

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, \dots, \quad \mathbb{E}[X] = \frac{1}{p}, \quad \text{Var}(X) = \frac{1 - p}{p^2}$$

- **Poisson**(λ): modeled count in a fixed window,

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots, \quad \mathbb{E}[X] = \text{Var}(X) = \lambda$$

Typical task: Recognize the model from the story, write down its PMF.

Review: Joint, marginal, and conditional PMFs

- Joint PMF:

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

- Marginals:

$$p_X(x) = \sum_y p_{X,Y}(x,y), \quad p_Y(y) = \sum_x p_{X,Y}(x,y)$$

- Conditional PMFs:

$$p_{X|Y}(x | y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \quad \text{if } p_Y(y) > 0$$

- Reconstruction (for $p_Y(y) > 0$):

$$p_{X,Y}(x,y) = p_Y(y) p_{X|Y}(x | y)$$

Typical task: Navigate between joint, marginal, and conditional PMFs; compute probabilities such as $P(X < Y)$, $P(X + Y = z)$, or $P(X = x | Y = y)$.

Review: Conditional expectation and law of total expectation

- Conditional expectation (for $p_Y(y) > 0$):

$$\mathbb{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

- Law of total expectation:

$$\mathbb{E}[X] = \sum_{y:p_Y(y)>0} \mathbb{E}[X \mid Y = y] p_Y(y).$$

More generally, if $\{A_i\}$ is a partition with $P(A_i) > 0$,

$$\mathbb{E}[X] = \sum_i P(A_i) \mathbb{E}[X \mid A_i].$$

Typical task: Compute a hard mean by conditioning on an easier variable, event, case, or state of the world.

Review: Independence of random variables

- Definition:

$$p_{X,Y}(x,y) = p_X(x)p_Y(y) \quad \text{for all } x,y$$

- If X and Y are independent, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y], \quad \mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)].$$

- If X and Y are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

- Cautions:

- linearity of expectation does *not* require independence;
- $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ alone does *not* imply independence.

Typical task: Check independence by testing joint-PMF factorization for all relevant (x,y) , then use independence to factor expectations or add variances.

Review: Continuous random variables, PDFs, and CDFs

For a random variable with PDF f_X ,

$$f_X(x) \geq 0, \quad \int_{-\infty}^{\infty} f_X(x) dx = 1, \quad P(X = x) = 0.$$

CDF and interval probabilities:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt,$$

$$P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x) dx.$$

The CDF difference works for any random variable; the integral requires a PDF.

Recovering the density: $f_X(x) = F'_X(x)$ wherever f_X is continuous.

Warning: A density value is not a probability and can exceed 1.

Typical task: Identify support, normalize the PDF, and integrate over events.

Review: Uniform and exponential models

Uniform: $X \sim \text{Uniform}(a, b)$, where $a < b$

$$f_X(x) = \frac{1}{b-a} \quad (a \leq x \leq b), \quad F_X(x) = \begin{cases} 0, & x < a, \\ (x-a)/(b-a), & a \leq x \leq b, \\ 1, & x > b. \end{cases}$$

Exponential: $X \sim \text{Exponential}(\lambda)$, with rate $\lambda > 0$

$$f_X(x) = \lambda e^{-\lambda x} \quad (x \geq 0), \quad F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

Both PDFs are zero outside their supports. For $t \geq 0$, $P(X > t) = e^{-\lambda t}$ in the exponential model.

Interpretation: Uniform probabilities are interval-length ratios; exponential waiting times arise between arrivals in a constant-rate Poisson model.

Typical task: Use the support and CDF to compute an interval or tail probability.

Review: Continuous expectation and variance

Replace probability-weighted sums by integrals:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf_X(x) dx, \quad \mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) dx.$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

The affine-transformation rules for mean and variance still apply.

Useful model summaries:

Model	$\mathbb{E}[X]$	$\text{Var}(X)$
Uniform(a, b)	$(a + b)/2$	$(b - a)^2/12$
Exponential(λ)	$1/\lambda$	$1/\lambda^2$

Check: For $X \sim \text{Uniform}(0, 2)$, $\mathbb{E}[X] = 1$ and $\text{Var}(X) = 1/3$.

Typical task: Integrate over the support to find $\mathbb{E}[X]$, $\mathbb{E}[g(X)]$, or $\mathbb{E}[X^2]$.

Review: Normal probabilities and quantiles

Normal model: $X \sim N(\mu, \sigma^2)$: $\mathbb{E}[X] = \mu$, $\text{Var}(X) = \sigma^2$, and standard deviation σ .

Standardization:

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1), \quad \Phi(z) = P(Z \leq z).$$

$$P(a < X \leq b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right).$$

Symmetry: $\Phi(-z) = 1 - \Phi(z)$. Right tail: $P(Z > z) = 1 - \Phi(z)$.

Quantiles: For $0 < q < 1$,

$$P(X \leq x_q) = q \quad \Longleftrightarrow \quad x_q = \mu + \sigma\Phi^{-1}(q).$$

A 95th percentile uses $z_{0.95} \approx 1.645$; a central 95% interval is approximately $\mu \pm 1.96\sigma$.

Typical task: Find a probability from a cutoff, or find a cutoff from a probability.

Final checklist / common pitfalls

- Define the sample space, events, and random variables before computing.
- Identify the tool:
 - Counting / conditioning / Bayes' rule / independence
 - PMFs / PDFs / CDFs / moments / normal probabilities and quantiles
- Distinguish possibly confusing concepts carefully:
 - Disjoint vs. independent
 - Pairwise vs. mutual independence
 - Joint vs. marginal vs. conditional PMFs
 - Density vs. probability; variance vs. standard deviation
 - Left-tail vs. right-tail probability; percentile vs. central interval
- Remember:
 - Order matters in permutations, not in combinations
 - $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$ in general
 - Linearity of expectation does *not* require independence
 - Independence guarantees $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$
- Justify why a formula applies: partition, disjointness, equal likelihood, or independence.

Wrap-up: How to prepare

Use the practice materials for review

- First attempt them under exam conditions, using only the permitted aids under time pressure.
- Mark which steps failed: identifying the tool, setting up the model, or doing the calculation.
- Then review and prepare your note sheet around those weak points.

When preparing your note sheet

- Include definitions and assumptions, not only formulas.
- Ensure you are fluent with the tools you have written.

During the exam

- Succinctly write down the relevant event or random variable before using a formula.
- Keep notation clear: distinguish events, values, PMFs, PDFs, CDFs, and expectations.
- If stuck, draw the event/support, condition on a useful case, or try the complement.

Advice: Be fluent in translating words to probabilistic objects and choosing the right tool.