

STA 130A: Mathematical Statistics: Brief Course

Lecture 12: Joint Continuous Distributions

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Announcements

Office hours

- Instructor office hours today (Wed, Oct 21) are **canceled**; please post questions on Piazza

Midterm grading policy

- Midterm 1 exam and solutions are available on the course webpage
- Only the **higher of your two midterm scores** will count; the lower score will be dropped
- Use Midterm 1 to identify gaps; attend lectures and practice to prepare for Midterm 2

Coming up

- The lecture on Fri, Oct 23 will also be held on Zoom; see Canvas for the link
- Homework 4 will be released Fri, Oct 23 and is due Thu, Oct 29, 11:59 PM PT

Agenda

Before Midterm 1, we studied:

- Joint, marginal, and conditional PMFs
- Continuous PDFs, CDFs, and normal models

Today: from one continuous random variable to several

- Joint PDFs of two continuous random variables
- Marginal PDFs
- Joint CDFs
- Expectations with joint PDFs
- A three-variable example: probabilities and marginals

Recap: One continuous random variable

A random variable X is continuous with PDF f_X if

$$P(X \in B) = \int_B f_X(x) dx.$$

- A valid PDF satisfies

$$f_X(x) \geq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} f_X(x) dx = 1.$$

- For intervals:

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx.$$

- Expectation (when finite):

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

The cumulative distribution function (CDF) of X is

$$F_X(x) = \int_{-\infty}^x f_X(t) dt.$$

Jointly continuous random variables

Definition (Joint PDF)

Random variables X and Y are **jointly continuous** if there exists a nonnegative function $f_{X,Y}$ such that

$$P((X, Y) \in A) = \iint_A f_{X,Y}(x, y) \, dx \, dy$$

for regions $A \subseteq \mathbb{R}^2$. The function $f_{X,Y}$ is called the **joint PDF** of X and Y .

A valid joint PDF satisfies

$$f_{X,Y}(x, y) \geq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = 1.$$

Joint probability as volume under a surface

For jointly continuous X, Y ,

$$P((X, Y) \in A) = \iint_A f_{X,Y}(x, y) dx dy.$$

Interpretation:

- The joint PDF is a density surface over the (x, y) -plane.
- Probability is volume under the surface over the region A .
- A value $f_{X,Y}(x, y)$ is not itself a probability.

For rectangles:

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y) dy dx.$$

Not every continuous pair is jointly continuous

Even if X and Y are continuous, the pair (X, Y) need not be jointly continuous.

Example

Let X be continuous, for example $X \sim \text{Uniform}(0, 1)$, and let

$$Y = X.$$

Then both X and Y are continuous marginally, but

$$P(Y = X) = 1.$$

A joint PDF over the plane cannot place positive probability on a single line, because a line has zero area.

$\implies (X, Y)$ is not jointly continuous, although X and Y are individually continuous.

Marginal PDFs

Given a joint PDF $f_{X,Y}$, the marginal PDFs are obtained by integrating out the other variable:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx.$$

This is the continuous analogue of summing out a variable from a joint PMF:

$$p_X(x) = \sum_y p_{X,Y}(x, y).$$

Key idea:

discrete sums $\longleftrightarrow$ continuous integrals.

Example: Uniform density on a triangle (1/2)

Example

Let (X, Y) be uniformly distributed over the triangle

$$T = \{(x, y) : 0 \leq y \leq x \leq 1\}.$$

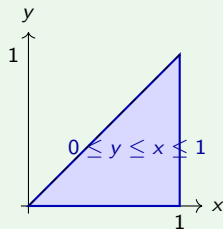
Thus,

$$f_{X,Y}(x, y) = \begin{cases} c, & \text{if } (x, y) \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Since T has area $1/2$, normalization gives $c/2 = 1$, so $c = 2$.

Therefore,

$$f_{X,Y}(x, y) = \begin{cases} 2, & 0 \leq y \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$



Example: Uniform density on a triangle (2/2)

Example

Next, compute the marginal PDFs from the joint PDF

$$f_{X,Y}(x,y) = 2, \quad 0 \leq y \leq x \leq 1.$$

We find f_X by fixing x and integrating over the allowed values $0 \leq y \leq x$:

$$f_X(x) = \int_0^x 2 \, dy = 2x, \quad 0 \leq x \leq 1.$$

We find f_Y by fixing y . Since $0 \leq y \leq x \leq 1$, the allowed values of x are $y \leq x \leq 1$:

$$f_Y(y) = \int_y^1 2 \, dx = 2(1 - y), \quad 0 \leq y \leq 1.$$

Both marginal densities are zero outside $[0, 1]$.

Note: A uniform joint density need not give uniform marginals

Pop-up quiz

For the triangular density

$$f_{X,Y}(x,y) = 2, \quad 0 \leq y \leq x \leq 1,$$

which expression gives $P(X + Y \leq 1)$?

A) $\int_0^1 \int_0^x 2 \, dy \, dx$

B) $\int_0^{1/2} \int_0^x 2 \, dy \, dx + \int_{1/2}^1 \int_0^{1-x} 2 \, dy \, dx$

C) $\int_0^1 \int_0^{1-y} 2 \, dx \, dy$

D) $\int_0^1 \int_0^{1-x} 2 \, dy \, dx$

Answer: B. For $0 \leq x \leq 1/2$, the triangle constraint $y \leq x$ is tighter; for $1/2 \leq x \leq 1$, the event constraint $y \leq 1 - x$ is tighter.

Check: The two integrals are $1/4$ each, so the probability is $1/2$.

Self-study: Buffon's needle (1/2)

Example (Buffon's needle)

Parallel lines are spaced distance d apart. A needle of length $0 < \ell \leq d$ is dropped randomly.

Let

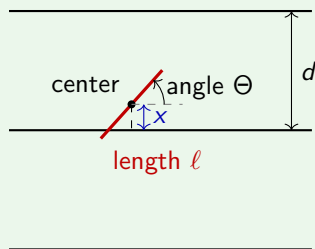
X = center-to-nearest-line distance,

Θ = acute angle with the parallel lines.

Assume

$$X \sim \text{Uniform}(0, d/2), \quad \Theta \sim \text{Uniform}(0, \pi/2),$$

and X, Θ are independent.



The perpendicular half-length of the needle is $(\ell/2) \sin \Theta$, and it crosses a line when

$$X \leq \frac{\ell}{2} \sin \Theta.$$

Self-study: Buffon's needle (2/2)

Example (Buffon's needle)

By uniformity and independence, the joint PDF of X and Θ is constant:

$$f_{X,\Theta}(x, \theta) = \begin{cases} \frac{1}{(d/2)(\pi/2)} = \frac{4}{\pi d}, & \forall (x, \theta) \in [0, d/2] \times [0, \pi/2], \\ 0, & \text{otherwise.} \end{cases}$$

Since $\ell \leq d$, we have $(\ell/2) \sin \theta \leq d/2$, so the crossing probability is

$$\begin{aligned} P(\text{cross}) &= \int_0^{\pi/2} \int_0^{(\ell/2) \sin \theta} \frac{4}{\pi d} dx d\theta \\ &= \frac{4}{\pi d} \int_0^{\pi/2} \frac{\ell}{2} \sin \theta d\theta \\ &= \frac{2\ell}{\pi d}. \end{aligned}$$

This gives one classical method for empirically estimating π .

Joint cumulative distribution function

Definition

The **joint cumulative distribution function** of two random variables X and Y is

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y).$$

For jointly continuous random variables,

$$F_{X,Y}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(s, t) dt ds.$$

Where $F_{X,Y}$ is twice continuously differentiable, a version of the joint density is

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y).$$

Example: Joint CDF of uniform random variables

Example

Let $f_{X,Y}(x, y) = 1$ on $[0, 1]^2$, and zero elsewhere.

$$F_{X,Y}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(s, t) dt ds = \begin{cases} 0, & x < 0 \text{ or } y < 0, \\ xy, & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ y, & x \geq 1, 0 \leq y \leq 1, \\ x, & 0 \leq x \leq 1, y \geq 1, \\ 1, & x \geq 1, y \geq 1. \end{cases}$$

On the interior of the unit square,

$$\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) = 1 = f_{X,Y}(x, y).$$

Expectation with a joint PDF

For jointly continuous X, Y , assume all expectations below are finite.

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X, Y}(x, y) dx dy.$$

In particular,

$$\mathbb{E}[X] = \iint x f_{X, Y}(x, y) dx dy, \quad \mathbb{E}[Y] = \iint y f_{X, Y}(x, y) dx dy.$$

Linearity of expectation still holds:

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c.$$

- Recall no independence is required for linearity of expectation.

Example: Triangle density

Example

For the triangular density

$$f_{X,Y}(x,y) = 2, \quad 0 \leq y \leq x \leq 1,$$

compute $P(Y \leq X/2)$, $\mathbb{E}[X]$, and $\mathbb{E}[Y]$.

For each fixed $x \in [0, 1]$, the original triangle allows $0 \leq y \leq x$. The event $Y \leq X/2$ further restricts this to $0 \leq y \leq x/2$. Thus

$$P(Y \leq X/2) = \int_0^1 \int_0^{x/2} 2 \, dy \, dx = \int_0^1 x \, dx = \frac{1}{2}.$$

Also,

$$\mathbb{E}[X] = \int_0^1 \int_0^x 2x \, dy \, dx = \int_0^1 2x^2 \, dx = \frac{2}{3},$$

$$\mathbb{E}[Y] = \int_0^1 \int_0^x 2y \, dy \, dx = \int_0^1 x^2 \, dx = \frac{1}{3}.$$

Self-study check: Probability on a line

Suppose X, Y are jointly continuous with joint PDF $f_{X,Y}$. What is $P(X = Y)$?

- A) Always 1
- B) Always 0
- C) Equal to $\int f_{X,Y}(x, x) dx$
- D) Cannot be determined from the joint PDF

Answer: B.

The event $\{X = Y\}$ lies on a line in the plane, which has area zero under a joint PDF.

Follow-up: Recall the earlier example $Y = X$ and explain why it cannot have a joint PDF.

Reference: More than two continuous random variables

$X_1, \dots, X_n$ are jointly continuous if there is a nonnegative function (joint PDF)

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n)$$

whose integrals give probabilities, as below. In particular,

$$\int \cdots \int f_{X_1, \dots, X_n}(x_1, \dots, x_n) dx_1 \cdots dx_n = 1.$$

- Probabilities are computed by integrating over regions:

$$P((X_1, \dots, X_n) \in A) = \int_A f_{X_1, \dots, X_n}(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

- Marginals are obtained by integrating out the variables not of interest; for example,

$$f_{X_1}(x_1) = \int \cdots \int f_{X_1, \dots, X_n}(x_1, \dots, x_n) dx_2 \cdots dx_n.$$

Example: Three continuous random variables

Example

Let (X, Y, Z) have joint PDF

$$f_{X,Y,Z}(x,y,z) = \begin{cases} 3, & 0 \leq x \leq 1, \quad 0 \leq y \leq x, \quad 0 \leq z \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

At fixed x , the (y, z) -slice is a square of area x^2 ; normalization gives $\int_0^1 3x^2 dx = 1$.

Marginals: integrate out the coordinates not retained.

$$f_{X,Y}(x,y) = \int_0^x 3 dz = 3x, \quad 0 \leq y \leq x \leq 1,$$

$$f_X(x) = \int_0^x 3x dy = 3x^2, \quad 0 \leq x \leq 1.$$

Probability: restrict the integration region.

$$P\left(X \leq \frac{1}{2}\right) = \int_0^{1/2} \int_0^x \int_0^x 3 dz dy dx = \int_0^{1/2} 3x^2 dx = \frac{1}{8}.$$

Wrap-up

X, Y are **jointly continuous** with **joint PDF** $f_{X,Y}$ if

$$P((X, Y) \in A) = \iint_A f_{X,Y}(x, y) dx dy.$$

- A joint PDF gives probabilities by integration over regions, not by its point values.
- **Marginals** are obtained by integrating out the other variables:

$$f_X(x) = \int f_{X,Y}(x, y) dy.$$

- **Expectations** are weighted averages over the joint density:

$$\mathbb{E}[g(X, Y)] = \iint g(x, y) f_{X,Y}(x, y) dx dy.$$

Caution: Continuous marginals need not give a joint PDF. Always identify the integration region.

Suggested reading: [BT08, Sec. 3.4]

Next lecture: Continuous conditioning, conditional expectation, and independence.

References



Dimitri Bertsekas and John N Tsitsiklis.

Introduction to probability, volume 1.

Athena Scientific, 2nd edition, 2008.