

STA 130A: Mathematical Statistics: Brief Course

Lecture 14: Derived Distributions

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Agenda

So far: Continuous random variables

- Continuous random variable and PDF
- Cumulative distribution function
- Normal random variable
- Jointly continuous random variables
- Conditioning

Today: Derived distributions

- Functions of one random variable: $Z = g(X)$
 - Generic method: support $\rightarrow$ CDF $\rightarrow$ PDF
 - Monotone transformation formula
 - Affine transformations as a special case
- Functions of two random variables: $Z = g(X, Y)$
 - CDF by integrating over a region
 - Sum of independent random variables: convolution

Recap: Continuous random variable

A random variable X is continuous if it has a PDF f_X :

$$P(X \in A) = \int_A f_X(x) dx.$$

Its CDF satisfies

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt, \quad f_X(x) = F'_X(x) \quad \text{where } f_X \text{ is continuous.}$$

A pair of random variables X, Y is jointly continuous if $\exists f_{X,Y}$ such that

$$P((X, Y) \in A) = \iint_A f_{X,Y}(x, y) dx dy.$$

Question for today: If

$$Z = g(X) \quad \text{or} \quad Z = g(X, Y),$$

how do we find the support, CDF, and PDF of Z ?

Distribution of a function of a random variable

Let X be a random variable and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Let

$$Z = g(X).$$

- When X is discrete, we computed the PMF of Z by

$$p_Z(z) = \sum_{x:g(x)=z} p_X(x).$$

- When X is continuous, $P(X = x) = 0$ for all x ; can we sum $f_X(x)$ over x s.t. $g(x) = z$? **No!**

Generic, CDF-first workflow for $Z = g(X)$

1. Identify the support: possible values of $Z = g(X)$.
2. Compute the CDF:

$$F_Z(z) = P(Z \leq z) = P(g(X) \leq z).$$

3. Differentiate on the support:

$$f_Z(z) = \frac{d}{dz} F_Z(z).$$

Example: A quadratic transformation

Example

Let $X \sim \text{Uniform}(-1, 1)$, and let

$$Z = X^2.$$

Question: Find the PDF of Z .

- Observe that Z is supported on $[0, 1] \rightarrow$ for $z < 0$, $F_Z(z) = 0$; and for $z \geq 1$, $F_Z(z) = 1$.
- For $0 \leq z < 1$,

$$F_Z(z) = P(Z \leq z) = P(X^2 \leq z) = P(-\sqrt{z} \leq X \leq \sqrt{z}) = \frac{2\sqrt{z}}{2} = \sqrt{z}.$$

Thus,

$$F_Z(z) = \begin{cases} 0, & z < 0, \\ \sqrt{z}, & 0 \leq z < 1, \\ 1, & z \geq 1, \end{cases} \quad \text{and therefore} \quad f_Z(z) = F'_Z(z) = \begin{cases} \frac{1}{2\sqrt{z}}, & 0 < z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Check: $\int_{\mathbb{R}} f_Z(z) dz = \int_0^1 \frac{1}{2\sqrt{z}} dz = 1.$

Example: An exponential transformation

Example

Let $X \sim \text{Uniform}(0, 1)$, and let

$$Z = e^X.$$

Question: Find the PDF of Z .

The support of $Z = e^X$ is $[1, e]$; endpoint values have probability zero.

For $1 < z < e$,

$$F_Z(z) = P(Z \leq z) = P(e^X \leq z) = P(X \leq \log z) = \log z.$$

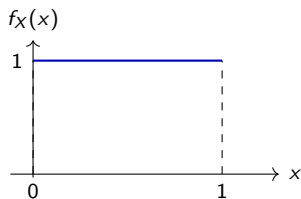
Thus,

$$F_Z(z) = \begin{cases} 0, & z \leq 1, \\ \log z, & 1 < z < e, \\ 1, & z \geq e, \end{cases} \quad \text{and therefore} \quad f_Z(z) = \begin{cases} \frac{d}{dz} \log z = \frac{1}{z}, & 1 < z < e, \\ 0, & \text{otherwise.} \end{cases}$$

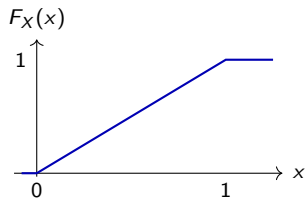
Interpretation: More stretching means less density; here $f_Z(z) = 1/z$.

Visual illustration: $Z = e^X$

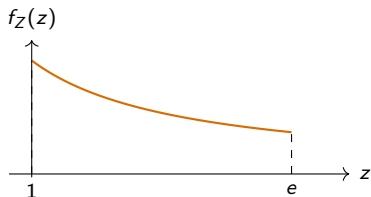
PDF of $X \sim \text{Uniform}(0, 1)$



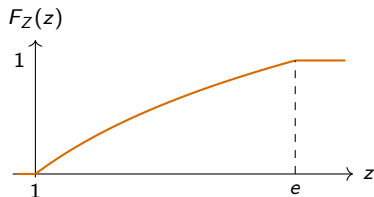
CDF of X



PDF of $Z = e^X$



CDF of $Z = e^X$



Message: A transformation may change both the support and the shape of the density.

PDF formula for a strictly monotone transformation

Definition (Strictly monotone on the support)

A function g is strictly monotone on the support of X if it is either strictly increasing or strictly decreasing there.

A strictly monotone function g is injective (one-to-one), so it has an inverse.

PDF formula for a smooth monotone transformation

Suppose X has a PDF on an interval I , and g is continuously differentiable and strictly monotone on its interior, with $g'(x) \neq 0$ there. Write

$$x = h(z) = g^{-1}(z).$$

For $Z = g(X)$, a density on the transformed interior is

$$f_Z(z) = f_X(h(z)) \cdot |h'(z)|.$$

Outside the support, $f_Z(z) = 0$.

Why the monotone formula works

Proof: Why does this formula hold?

1. Assume g is strictly increasing, and let $h = g^{-1}$. Then

$$F_Z(z) = P(Z \leq z) = P(g(X) \leq z) = P(X \leq h(z)) = F_X(h(z)).$$

Differentiating with respect to z , we obtain

$$f_Z(z) = F'_Z(z) = f_X(h(z)) h'(z).$$

2. If g is strictly decreasing, the inequality reverses:

$$F_Z(z) = P(X \geq h(z)) = 1 - F_X(h(z)),$$

where the endpoint has probability 0 because X is continuous. Differentiating gives

$$f_Z(z) = -f_X(h(z)) h'(z) = f_X(h(z)) |h'(z)|.$$

Interpretation

- $h'(z)$ accounts for the stretching/compression of intervals.
- The absolute value ensures the density remains nonnegative.

Example: Revisit $Z = e^X$

Example

Let $X \sim \text{Uniform}(0, 1)$, and let

$$Z = e^X.$$

Here

$$z = e^x \iff x = h(z) = \log z.$$

Observe that

$$h'(z) = \frac{1}{z}.$$

Since $f_X(x) = 1$ for $0 < x < 1$, we must have $0 < \log z < 1$, i.e., $1 < z < e$. Thus,

$$f_Z(z) = f_X(\log z) \left| \frac{1}{z} \right| = 1 \cdot \frac{1}{z} = \frac{1}{z}, \quad 1 < z < e.$$

This result agrees with the CDF-first calculation.

Self-study: Reciprocal transformation

Example

Let $X \sim \text{Uniform}(1, 2)$, and let

$$Z = \frac{1}{X}.$$

The transformation is strictly decreasing. Its inverse is

$$x = h(z) = \frac{1}{z}, \quad |h'(z)| = \frac{1}{z^2}.$$

Since $1 < X < 2$, the possible values of Z are $\frac{1}{2} < Z < 1$. Because $f_X(x) = 1$ for $1 < x < 2$,

$$f_Z(z) = f_X\left(\frac{1}{z}\right) \frac{1}{z^2} = \frac{1}{z^2}, \quad \frac{1}{2} < z < 1.$$

Thus,

$$f_Z(z) = \begin{cases} z^{-2}, & \frac{1}{2} < z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Special case: Affine transformations

Let's consider a special instance of strictly monotonic transforms.

Let X have a PDF and

$$Z = aX + b, \quad a \neq 0.$$

Then

$$x = h(z) = \frac{z - b}{a}, \quad |h'(z)| = \frac{1}{|a|}.$$

Thus,

$$f_Z(z) = \frac{1}{|a|} f_X\left(\frac{z - b}{a}\right).$$

Interpretation:

- b shifts the distribution.
- a stretches or compresses the scale.
- The factor $1/|a|$ preserves total area under f_Z to remain 1.

Example: Affine transformation of a uniform random variable

Example

Let $X \sim \text{Uniform}(0, 1)$, and let

$$Z = 2X + 3.$$

Question: Identify the support and the distribution of Z .

Let $g : x \mapsto 2x + 3$ and let $h = g^{-1}$. Then

$$h(z) = \frac{z-3}{2}, \quad |h'(z)| = \frac{1}{2}.$$

Since $f_X(x) = 1$ for $0 < x < 1$, the support of Z is $[3, 5]$, and

$$f_Z(z) = \begin{cases} f_X\left(\frac{z-3}{2}\right) \cdot \frac{1}{2} = \frac{1}{2}, & 3 < z < 5, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$Z \sim \text{Uniform}(3, 5).$$

Pop-up quiz: Transformation

Let $X \sim \text{Uniform}(0, 1)$, and let

$$Z = 2 - X.$$

Which density and interval where it is positive are correct for Z ?

- A) $1 < z < 2$, $f_Z(z) = 1$
- B) $0 < z < 1$, $f_Z(z) = 1$
- C) $1 < z < 2$, $f_Z(z) = 2 - z$
- D) $0 < z < 2$, $f_Z(z) = \frac{1}{2}$

Answer: A.

Here $x = 2 - z$, so $|dx/dz| = 1$. Also,

$$0 < x < 1 \iff 1 < z < 2.$$

Thus $f_Z(z) = 1$ for $1 < z < 2$.

Follow-up: For $W = 3X - 2$, the support is $[-2, 1]$; what is its density?

Function of two random variables

Let (X, Y) be jointly continuous, and let

$$Z = g(X, Y).$$

To find the distribution of Z , a general strategy is again to identify the CDF first:

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(g(X, Y) \leq z) \\ &= \iint_{\{(x,y): g(x,y) \leq z\}} f_{X,Y}(x, y) \, dx \, dy. \end{aligned}$$

If Z has a PDF, recover it from the CDF where differentiation is valid:

$$f_Z(z) = F'_Z(z).$$

Main challenge: Identifying the integration region correctly.

Practical rules: Draw the region $\{(x, y) : g(x, y) \leq z\}$. Split the integral as needed whenever the boundary changes.

Self-study: Product of two independent uniforms (1/2)

Example

Let (X, Y) be uniformly distributed on the unit square $[0, 1]^2$, and let

$$Z = XY.$$

Question: Find the CDF and the PDF of Z .

For $0 < z < 1$, $F_Z(z) = P(XY \leq z)$. For a fixed $x \in (0, 1)$,

$$XY \leq z \iff Y \leq \frac{z}{x}.$$

Geometrically, this is the area below the curve $y = z/x$ inside the unit square.

- If $0 < x \leq z$, then $z/x \geq 1$, so all $0 < Y < 1$ are allowed.
- If $z < x < 1$, then $0 < Y \leq z/x$.

$$F_Z(z) = \int_0^z 1 \, dx + \int_z^1 \frac{z}{x} \, dx = z + z \log(1/z), \quad 0 < z < 1.$$

Self-study: Product of two independent uniforms (2/2)

Example

For $Z = XY$ with $(X, Y) \sim \text{Uniform}([0, 1]^2)$, we found

$$F_Z(z) = \begin{cases} 0, & z \leq 0, \\ z + z \log(1/z), & 0 < z < 1, \\ 1, & z \geq 1. \end{cases}$$

Differentiating F_Z with respect to z ,

$$f_Z(z) = \frac{d}{dz} \left[z + z \log\left(\frac{1}{z}\right) \right] = \log\left(\frac{1}{z}\right), \quad 0 < z < 1.$$

Therefore,

$$f_Z(z) = \begin{cases} \log(1/z), & 0 < z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Takeaway: For $Z = g(X, Y)$, the key step is identifying the region $\{g(x, y) \leq z\}$.

Sum of independent random variables: Convolution

Let X, Y be independent continuous random variables and $Z = X + Y$.

Convolution formula

$$f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx.$$

- **Proof idea:** $f_{X,Z}(x, z) = f_{X,Y}(x, z - x) = f_X(x) f_Y(z - x)$ by independence
- **Support rule:** The integral only includes x 's for which both

$$x \in \text{supp}(X) \quad \text{and} \quad z - x \in \text{supp}(Y).$$

- **Discrete analogue:** $p_Z(z) = \sum_x p_X(x) p_Y(z - x)$.
- **Interpretation:** To get $X + Y = z$, sum over all possible ways $x + (z - x) = z$.

Example: Sum of two independent uniforms

Example

Let $X, Y \stackrel{\text{independent}}{\sim} \text{Uniform}(0, 1)$, and $Z = X + Y$.

Question: Find the PDF of Z .

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx.$$

Since $f_X(x) = 1$ for $0 < x < 1$ and $f_Y(z-x) = 1$ when $0 < z-x < 1$, the integrand $f_X(x)f_Y(z-x)$ is nonzero only when

$$0 < x < 1 \quad \text{and} \quad z-1 < x < z.$$

Thus x must lie in the interval

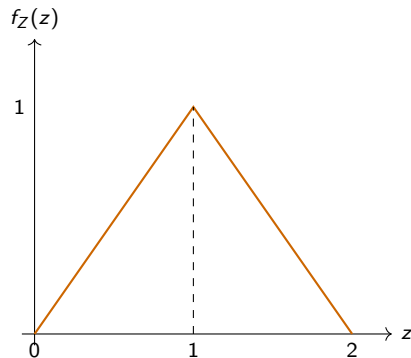
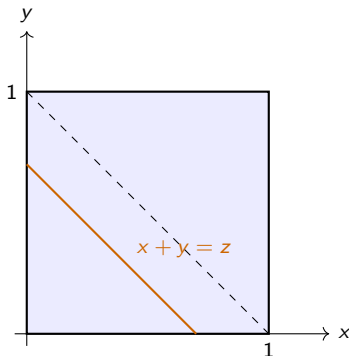
$$\max(0, z-1) < x < \min(1, z).$$

The density of Z at z is equal to the length of this interval (whenever it is positive):

$$f_Z(z) = \begin{cases} z, & 0 < z < 1, \\ 2-z, & 1 \leq z < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Geometric view of $X + Y$

For independent $X, Y \sim \text{Uniform}(0, 1)$, the joint density is uniform on the unit square.



For a thin band $z < X + Y < z + \Delta z$, probability is its area. The perpendicular width is $\Delta z / \sqrt{2}$, so the density is the diagonal slice length divided by $\sqrt{2}$.

The slice length increases for $0 < z < 1$ and decreases for $1 < z < 2$, producing a triangular density.

Self-study: Sum of independent exponentials

Example

Let X, Y be independent $\text{Exponential}(\lambda)$ random variables, and let

$$Z = X + Y.$$

Question: Find the PDF of Z .

Since $X, Y > 0$, we have $Z > 0$. Using convolution, for $z > 0$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\ &= \lambda^2 e^{-\lambda z} \int_0^z dx = \lambda^2 z e^{-\lambda z}. \end{aligned}$$

Thus,

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & z > 0, \\ 0, & z \leq 0. \end{cases}$$

Remark: This distribution is often called an Erlang/Gamma distribution with shape 2 and rate λ .

Self-study check: Difference of two uniforms

Let X, Y be independent $\text{Uniform}(0, 1)$, and let

$$Z = X - Y.$$

Which describes the density of Z correctly (zero off the stated interval)?

A) $f_Z(z) = \frac{1}{2}, \quad -1 < z < 1$

C) $f_Z(z) = 1, \quad -1 < z < 1$

B) $f_Z(z) = \begin{cases} z + 1, & -1 < z < 0, \\ -z + 1, & 0 \leq z < 1, \\ 0, & \text{otherwise} \end{cases}$

D) $f_Z(z) = z, \quad -1 < z < 1$

Answer: B. Since $Z = X - Y$, write $Y = X - z$. The convolution-type formula is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(x - z) dx.$$

The integrand is nonzero when $0 < x < 1$ and $0 < x - z < 1$, or equivalently,

$$\max(0, z) < x < \min(1, 1 + z).$$

Wrap-up

CDF first: Identify the support and rewrite the event:

$$F_Z(z) = P(g(X) \leq z).$$

If Z has a PDF, differentiate where valid. The CDF method also handles non-one-to-one transformations.

Smooth monotone transformations: Under the stated derivative conditions,

$$f_Z(z) = f_X(h(z))|h'(z)|, \quad h = g^{-1}.$$

Independent sums: For $Z = X + Y$,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x) dx.$$

The bounds must satisfy both support restrictions.

Suggested reading: [BT08, Sec. 4.1]

Next lecture: Covariance, correlation, and variance decomposition.

References



Dimitri Bertsekas and John N Tsitsiklis.

Introduction to probability, volume 1.

Athena Scientific, 2nd edition, 2008.