STA 130A: Mathematical Statistics: Brief Course

Lecture 24: Confidence Intervals for a Mean

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Announcements

Final exam: Mon, Dec 7, 3:30–5:30 PM PT

- **Location:** Wellman Hall 226 (our usual classroom)
- **Cumulative coverage:** Lectures 1–26
- Arrive early; the exam ends at **5:30 PM sharp**
- Permitted materials
  - **Notes: Three (3)** handwritten, letter-sized (8.5×11 inches), double-sided sheets
  - **Calculator:** A simple non-graphing scientific calculator
  - No other reference materials or electronic devices
- **SDC accommodations:** Confirm exam scheduling with AES online

Upcoming schedule

- **No class:** Wed, Nov 25 and Fri, Nov 27
- **Remote lecture:** Mon, Nov 30, 11:00–11:50 AM PT on Zoom

Clarification on grading

Course grading scheme: see the syllabus

- **Homework (30%):** Average of the best 6 of 7 assignments
- **Midterms (30%):** Higher of the two midterm scores
- **Final exam (40%)**
- **Participation credit:** Up to 3 percentage points added to the course score

Letter grades

- Baseline cutoffs: 97% for A+, 93% for A, 90% for A-, etc.
- Cutoffs may be adjusted **at quarter's end** based on relative standing, **only in students' favor**
- Individual exam and homework scores will not be curved

C- safety net

- If your grade under the scheme above is **D+ or below**, we also calculate an alternative score: **30% homework + 70% final exam + participation credit**
- If this score **exceeds 65 out of 100**, your course grade will be raised to **C-**

* Please do not request individual rankings or preliminary grade cutoffs.

* No other grading adjustments or individual exceptions will be made.

Agenda

Last time

- Sampling distributions and standard errors
- The sample mean and sample variance under normal sampling

Today: Quantifying uncertainty about a population mean

- Confidence intervals when σ is known
- Confidence intervals when σ is unknown
  - Student's t distribution
- Interpretation, assumptions, and interval width

Motivation: How precise is our estimate?

Example (Package weights)

A machine fills packages with an unknown population mean weight μ. Suppose a random sample of 16 packages gives

$$\bar{x} = 501.2 \text{ grams.}$$

Another sample would usually give a different sample mean.

- A point estimate gives one plausible value for μ.
- We also want to express the estimate's **sampling uncertainty**.

Question: How can we use a sampling distribution to construct a range that reliably covers the true mean?

We begin with a normal population and known σ, then allow σ to be unknown.

Recap: Standardizing the sample mean

Suppose $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, with $n \geq 2$ and $\sigma > 0$. Then

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right), \qquad Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

The denominator is the standard error:

$$\mathrm{SE}(\bar{X}) = \frac{\sigma}{\sqrt{n}}.$$

It describes how much the sample mean varies across repeated samples.

For $p \in (0,1)$, let z_p denote the standard normal quantile such that $\Phi(z_p) = P(Z \leq z_p) = p$. For example,

$$z_{0.975} \approx 1.96, \qquad P(-z_{0.975} \leq Z \leq z_{0.975}) = 0.95.$$

Confidence intervals with known σ

Under the normal sampling model, for $0 < \alpha < 1$,

$$P\left(-z_{1-\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{1-\alpha/2}\right) = 1 - \alpha.$$

Rearranging the same event gives

$$P\left(\bar{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

A $100(1-\alpha)\%$ confidence interval for μ

After observing the data, report

$$\bar{x} \ \pm \ z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Structure: point estimate $\pm$ critical value $\times$ standard error.

Example: A mean-weight interval with known σ

Example

Assume normal package weights with $\sigma = 4$ grams known. Suppose $n = 16$ and the observed sample mean is $\bar{x} = 501.2$ grams.

For a 95% confidence interval,

$$\mathrm{SE}(\bar{X}) = \frac{4}{\sqrt{16}} = 1 \text{ gram}, \qquad z_{0.975} \approx 1.96.$$

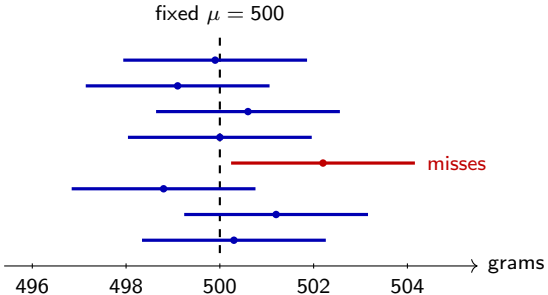
Thus the interval is

$$501.2 \ \pm \ z_{0.975} \cdot 1 \ \approx \ [499.24,\ 503.16] \text{ grams}.$$

Interpretation: We used a procedure that covers the population mean in 95% of repeated samples under this model.

What 95% confidence means

The population mean μ stays fixed. Each new sample produces a new interval.



Before sampling

$$P(L(X) \leq \mu \leq U(X)) = 0.95.$$

The endpoints $L(X)$ and $U(X)$ are random.

After sampling

The observed interval either contains μ or misses it.

Repeated-sampling meaning: Over many repetitions, about 95% of intervals cover μ.

Important: The target parameter μ is fixed; it is the interval that is random.

Pop-up quiz: Interpreting confidence

A 95% confidence interval for the population mean package weight is $[499.24, 503.16]$ grams.

Question: Which statement expresses the confidence level?

A) About 95% of individual packages weigh between these endpoints.

B) In repeated sampling, about 95% of intervals constructed this way cover μ.

C) The fixed unknown μ has probability 0.95 of lying in this observed interval.

D) About 95% of future sample means must fall in this observed interval.

Answer: B. The confidence level describes the procedure across repeated samples. The interval estimates a population mean, not the range of individual weights.

Sample size for a desired margin of error

Under normal sampling with known σ, the margin of error is

$$m = z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

To make $m \le d$, where $d > 0$, choose an integer n satisfying

$$n \ge \left(\frac{z_{1-\alpha/2}\sigma}{d}\right)^2.$$

Example (A margin of at most 1 gram)

For 95% confidence and known $\sigma = 4$ grams,

$$n \ge (4\, z_{0.975})^2 \approx 61.46.$$

Thus $n = 62$ packages suffice.

Planning rule: Round up to an integer. This calculation uses the normal model and the specified value of σ.

Motivation: What if the population SD is unknown?

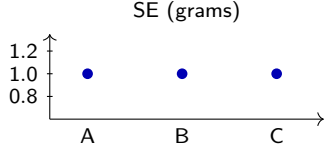
Example (Package weights)

Assume i.i.d. normal package weights. A sample of 16 packages gives $\bar{x} = 501.2$ grams.

So far: known $\sigma = 4$

$$\bar{x} \pm z_{0.975} \frac{\sigma}{\sqrt{n}} \approx 501.2 \pm 1.96.$$

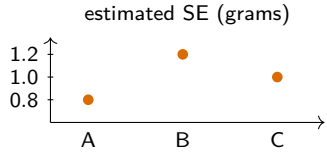
Fixed standard error: $\sigma/\sqrt{n} = 1$.



Now: unknown σ, observed $s = 4$

$$\bar{x} \pm z_{0.975} \frac{s}{\sqrt{n}} \approx 501.2 \pm 1.96.$$

Estimated standard error: $S/\sqrt{n}$.



A, B, C illustrate three repeated samples, each of size 16.

Question: Does replacing σ by s preserve 95% coverage?

Key change: With estimated SE, the standardized error is no longer exactly normal.

What changes when σ is unknown?

We estimate σ using the sample standard deviation

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}.$$

This changes the standardized quantity to

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}.$$

Example (Estimated standard error)

If $n = 16$ and the observed sample standard deviation is $s = 4$ grams, the estimated standard error is $s/\sqrt{n} = 1$ gram.

New issue: The denominator now varies across samples too. We need the distribution of this new ratio to choose the critical value.

Student's t distribution

Definition (Student's t distribution)

Let $Z \sim N(0,1)$ and $U \sim \chi^2_\nu$ be independent. Then

$$T = \frac{Z}{\sqrt{U/\nu}}$$

has a **Student's t distribution with ν degrees of freedom**, written $T \sim t_\nu$.

Intuition: A standard normal numerator is divided by a random scale.

- A small denominator can produce a large ratio.
- This creates heavier tails than those of a standard normal distribution.

Example: Student's t from a normal sample

Example

For i.i.d. $N(\mu, \sigma^2)$ observations, $n \geq 2$ and $\sigma > 0$, Lecture 23 gives

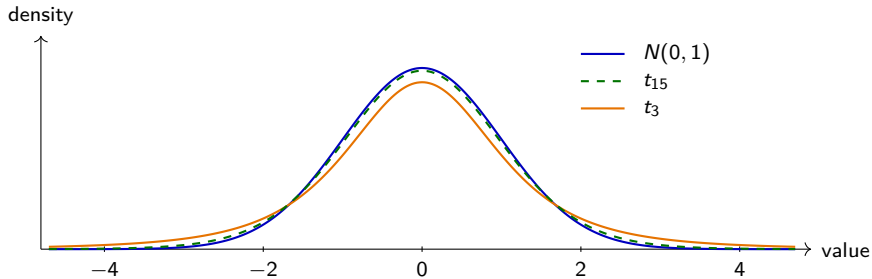
$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1), \qquad U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

Moreover, Z and U are independent because $\bar{X}$ and S^2 are independent.
Therefore,

$$\frac{Z}{\sqrt{U/(n-1)}} = \frac{(\bar{X} - \mu)/(\sigma/\sqrt{n})}{S/\sigma} = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

Key result: Replacing σ by S leads to t_{n-1} under normal sampling.

Normal and t densities



- Each density is symmetric about 0.
- Smaller degrees of freedom give heavier tails.
- As the degrees of freedom increase, the t distribution approaches $N(0,1)$.

Critical values from the t distribution

We use **lower-tail quantiles**:

$$P(T_\nu \le t_{\nu,p}) = p, \qquad T_\nu \sim t_\nu.$$

By symmetry,

$$P(-t_{\nu,1-\alpha/2} \le T_\nu \le t_{\nu,1-\alpha/2}) = 1 - \alpha.$$

Example (A 95% central interval with 15 degrees of freedom)

$$t_{15,0.975} \approx 2.131, \qquad P(-t_{15,0.975} \le T_{15} \le t_{15,0.975}) = 0.95.$$

Compare the critical values: 2.131 for t_{15}, versus 1.960 for $N(0,1)$.

Table lookup: Lower-tail probability 0.975 is upper-tail probability 0.025.

A confidence interval when σ is unknown

Theorem (Confidence interval for a normal mean)

For an i.i.d. normal sample of size $n \geq 2$, with $\sigma > 0$, a $100(1-\alpha)\%$ confidence interval for μ is

$$\bar{x} \pm t_{n-1,1-\alpha/2}\frac{s}{\sqrt{n}}.$$

The ingredients are

- **Point estimate:** $\bar{x}$
- **Estimated standard error:** $s/\sqrt{n}$
- **Critical value:** $t_{n-1,1-\alpha/2}$, using $n-1$ degrees of freedom

Why it works: The standardized error $(\bar{X}-\mu)/(S/\sqrt{n})$ has exactly the t_{n-1} distribution under this model.

Example: A mean-weight interval with unknown σ

Example

Assume normal package weights with unknown μ and σ. Suppose

$$n = 16, \qquad \bar{x} = 501.2 \text{ grams}, \qquad s = 4 \text{ grams}.$$

For a 95% interval,

$$\frac{s}{\sqrt{n}} = 1 \text{ gram}, \qquad n - 1 = 15, \qquad t_{15,0.975} \approx 2.131.$$

Therefore,

$$\bar{x} \pm t_{15,0.975} \frac{s}{\sqrt{n}} \approx 501.2 \pm 2.131 \approx [499.07,\ 503.33] \text{ grams}.$$

Here the estimated standard error is also 1 gram. The larger critical value gives a wider interval than when $\sigma = 4$ was known.

Exact and approximate mean intervals

All four cases use the form estimate $\pm$ critical value $\times$ SE.

Sampling model	σ	Critical value, SE	Coverage
i.i.d. normal	Known	$z_{1-\alpha/2}$, $\sigma/\sqrt{n}$	Exact
i.i.d. normal	Unknown	$t_{n-1,1-\alpha/2}$, $s/\sqrt{n}$	Exact
i.i.d., CLT approximation	Known	$z_{1-\alpha/2}$, $\sigma/\sqrt{n}$	Approximate
i.i.d., large-sample approximation	Unknown	$z_{1-\alpha/2}$, $s/\sqrt{n}$	Approximate

In the last row, using the t critical value with $s/\sqrt{n}$

Distinction: Exact t coverage comes from the normal sampling model.

What determines the interval's width?

$$\text{width} = 2 \times \text{critical value} \times \text{standard error}.$$

Example (Normal population with known $\sigma = 4$ grams)

Confidence level	n	Standard error	Interval width
95%	16	1 gram	≈ 3.92 grams
95%	64	0.5 gram	≈ 1.96 grams
99%	16	1 gram	≈ 5.15 grams

- Higher confidence requires a wider interval.
- More observations reduce the standard error.
- Greater population variability requires a wider interval at the same n.

With known σ and fixed confidence level, four times as many observations halve the width.

Pop-up quiz: Choosing and computing an interval

An i.i.d. normal sample has $n = 9$, $\bar{x} = 500$, and $s = 3$. The population standard deviation is unknown.

Question: Which is the 95% confidence interval for μ? Use $t_{8,0.975} \approx 2.306$.

A) 500 ± 1.960

B) 500 ± 2.306

C) 500 ± 6.918

D) 500 ± 0.769

Answer: B.

$$\frac{s}{\sqrt{n}} = \frac{3}{\sqrt{9}} = 1, \qquad n - 1 = 8, \qquad \bar{x} \pm t_{8,0.975} \frac{s}{\sqrt{n}} \approx 500 \pm 2.306.$$

Wrap-up

Confidence intervals for a normal population mean

$$\sigma \text{ known:} \quad \bar{x} \pm z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}},$$

$$\sigma \text{ unknown:} \quad \bar{x} \pm t_{n-1,1-\alpha/2}\frac{s}{\sqrt{n}}.$$

- A sampling distribution supplies the critical value.
- Confidence describes coverage across repeated samples.
- Distinguish exact normal-model coverage from large-sample approximation.

Suggested reading: [Ric07, Ch. 6 & 8].

Next lecture: Constructing point estimators by the method of moments.

References


John A Rice.
Mathematical statistics and data analysis, volume 9.
Thomson brooks/cole, Belmont, CA, 2007.