STA 131A Introduction to Probability Theory

Final Exam

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Name: ____________________ **Student ID:** ____________________

Instructions: This is a **closed-book final exam**. You may bring a pen or pencil, three letter-sized sheets of *hand-written* notes (both sides), and a *non-graphing* calculator. No other materials are allowed. **You have 120 minutes** to complete the exam. The **total score is 180 points**, with up to **10 bonus points**. A standard normal table is included at the end of the exam. Once you receive this exam problem set, please confirm that your copy includes all pages.

- Make sure to clearly write your name and student ID above.
- Present answers succinctly, but include all relevant steps for full credit.
- If you use a theorem, formula, or table value, make it clear.
- Partial credit is possible only if your reasoning is clearly shown and traceable by the grader.
- If necessary, round numerical answers to three decimal places.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Total	

Problem 1 (30 points in total).

(a) (6 points) Let A and B be events such that

$$P(A) = \frac{1}{2}, \qquad P(B) = \frac{2}{5}, \qquad P(A \cap B) = \frac{1}{10}.$$

Compute $P(A \cup B)$ and $P(A \mid B)$. Are A and B independent? Briefly justify your answer.

(b) (6 points) Let X have PMF

$$P(X = -1) = \frac{1}{4}, \qquad P(X = 0) = \frac{1}{2}, \qquad P(X = 2) = \frac{1}{4}.$$

Let $Y = X^2$. Find the PMF of Y, and compute $\mathbb{E}[Y]$.

(c) (6 points) Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 3x^2, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $P(X \ge 1/2)$ and $\mathbb{E}[X]$.

(d) (6 points) A fair die is rolled 4 times. How many ordered outcome sequences contain exactly one 6?

(e) (6 points) Suppose

$$\mathrm{Var}(X) = 9, \qquad \mathrm{Var}(Y) = 16, \qquad \mathrm{Cov}(X, Y) = -6.$$

Compute $\rho(X, Y)$ and $\mathrm{Var}(X + Y)$.

Problem 2 (30 points in total).

A hidden urn H is chosen uniformly from three possible urns A, B, C:

$$P(H = A) = P(H = B) = P(H = C) = \frac{1}{3}.$$

Each of the urns contains 6 balls whose compositions are

$$A : 4 \text{ red, } 2 \text{ blue}; \qquad B : 3 \text{ red, } 3 \text{ blue}; \qquad C : 1 \text{ red, } 5 \text{ blue.}$$

After the urn is chosen, two balls are drawn uniformly at random without replacement from that urn. Let

$$E = \{\text{exactly one red ball is drawn}\}.$$

(a) (10 points) Compute $P(E \mid H = A)$, $P(E \mid H = B)$, $P(E \mid H = C)$, and $P(E)$.

(b) (10 points) Compute the posterior odds

$$\frac{P(H = B \mid E)}{P(H = C \mid E)}, \qquad \frac{P(H = A \mid E)}{P(H = C \mid E)}.$$

Then use these ratios to identify the most likely state (A, B, or C) after observing E.

(c) **(10 points)** After observing E, one red and one blue ball have been removed from the selected urn, leaving four balls. Consider two actions:

- **Action 1:** earn 3 points for each red ball remaining and 1 point for each blue ball remaining.
- **Action 2:** earn 1 point for each red ball remaining and 2 points for each blue ball remaining.

The resulting state-dependent payoffs are summarized below:

	$H = A$	$H = B$	$H = C$
Action 1	10	8	4
Action 2	5	6	8

Which action (Action 1 or Action 2) has the larger posterior expected payoff given E?

(*Hint:* Since $P(E)$ is common to both posterior expected payoffs, you may compare unnormalized posterior-weighted payoffs.)

Problem 3 (30 points in total).

Let $G \sim \text{Bernoulli}(1/2)$. Conditional on G, the random variable X has the following distribution:

$$p_{X|G}(x \mid 0) = \begin{cases} 1/2, & \text{if } x \in \{-1, 1\}, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad p_{X|G}(x \mid 1) = \begin{cases} 1/2, & \text{if } x \in \{-2, 2\}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (10 points) Find the marginal PMF of X, and compute $\mathbb{E}[X]$.

(b) (10 points) Compute $\text{Var}(X)$. (*Hint:* Use the law of total variance.)

(c) (10 points) Compute $\text{Cov}(X, G)$, and determine whether X and G are independent. Briefly justify.

Problem 4 (30 points in total).

A task starts at time $X \in [0,1]$ and finishes at time Y. The task duration is $Y - X$. Suppose (X,Y) is uniformly distributed over the parallelogram

$$R = \{(x,y) : 0 \le x \le 1,\ x \le y \le x+1\}.$$

That is, there exists a constant $c \in \mathbb{R}$ such that

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (10 points) Sketch R, find the value of c, and determine whether X and Y are independent.

(b) (10 points) Find the marginal PDF $f_Y(y)$. Then compute $\mathbb{E}[X \mid Y = y]$ for $0 < y < 2$.

(c) (10 points) Let

$$D = Y - X$$

be a random variable representing the task duration. Find the CDF and PDF of D.

Problem 5 (30 points in total + 5 bonus points).

(a) (10 points) Let $X \sim \text{Exponential}(1/2)$, where $1/2$ is the rate parameter:

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Compute $P(X \geq 6)$. Then use Markov's and Chebyshev's inequalities to give two upper bounds for $P(X \geq 6)$. (*Hint:* You may use $\mathbb{E}[X] = 2$ and $\text{Var}(X) = 4$.)

(b) (10 points) Suppose $X_1, \ldots, X_{64}$ are independent response times, each identically distributed as Exponential(4), where 4 is the rate parameter. Let

$$\bar{X}_{64} = \frac{1}{64} \sum_{i=1}^{64} X_i.$$

You may use $\mathbb{E}[X_i] = 1/4$ and $\text{Var}(X_i) = 1/16$. Use the CLT to approximate

$$P(0.20 \leq \bar{X}_{64} \leq 0.30).$$

Express your answer using Φ, then give a numerical approximation.

(c) (10 points) Suppose n independent sensors produce calibration scores

$$U_1, \ldots, U_n \overset{\text{iid}}{\sim} \text{Uniform}(0, 1),$$

and let

$$M_n = \max\{U_1, \ldots, U_n\}$$

be the best calibration score. For $0 \le x \le 1$, compute $P(M_n \le x)$. Then determine whether

$$M_n \xrightarrow{P} 1.$$

(d*) (5 bonus points) Continue with $M_n = \max\{U_1, \ldots, U_n\}$ from part **(c)**. For a fixed constant $t > 0$, compute

$$\lim_{n\to\infty} P\big(n(1 - M_n) > t\big).$$

Problem 6 (30 points in total + 5 bonus points).

Consider the following measurement model for an unknown signal:

$$Y = \Theta + \varepsilon, \qquad \Theta \sim \text{Uniform}(0, 2), \qquad f_\varepsilon(e) = \begin{cases} 1 - |e|, & |e| \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here Θ is the signal we want to estimate, and ε is measurement noise independent of Θ. Equivalently,

$$f_{Y|\Theta}(y \mid \theta) = \begin{cases} 1 - |y - \theta|, & |y - \theta| \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) **(10 points)** Compute $\mathbb{E}[Y]$ and $\text{Var}(Y)$.
(*Hint:* You may use $\mathbb{E}[\varepsilon] = 0$, $\text{Var}(\varepsilon) = 1/6$, and condition on Θ.)

(b) **(10 points)** After observing $Y = 1.5$, use Bayes' rule to find the posterior density $f_{\Theta|Y}(\theta \mid 1.5)$. Then compute

$$P(\Theta \ge 1 \mid Y = 1.5).$$

(*Hint:* First identify the interval of θ-values for which $f_{Y|\Theta}(1.5 \mid \theta) > 0$.)

(c) **(10 points)** Suppose now that $\Theta = \theta$ is fixed, and we collect independent measurements

$$Y_i = \theta + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where the ε_i's are i.i.d. copies of ε. Let

$$\bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

Using the CLT approximation, find the smallest integer n such that the CLT approximation gives

$$P(|\bar{Y}_n - \theta| \leq 0.05 \mid \Theta = \theta) \geq 0.95.$$

(d*) **(5 bonus points)** After observing $Y = 1.5$, is $\mathbb{E}[\Theta \mid Y = 1.5]$ less than, equal to, or greater than 1.5? Briefly justify your answer either geometrically or by explicit computation.

Standard Normal Table

Entries give $\Phi(z) = P(Z \le z)$ for $Z \sim N(0,1)$. For negative values, use $\Phi(-z) = 1 - \Phi(z)$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990