STA 131A Introduction to Probability Theory

(Practice Final Exam – Version A)

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Name: ______________________ **Student ID:** __________________

Instructions. This final exam is **closed-book**, except for the permitted note sheets described below. You may use a pen or pencil, three letter-sized sheets of *hand-written* notes (both sides), and a simple non-graphing calculator; no other materials are allowed. You have 2 hours to complete the exam. A standard normal table is included on the last page. The **total score is 180 points**.

- Make sure to clearly write your name and student ID above.
- Present your answers clearly and show enough work to justify your conclusions for full credit.
- Partial credit is possible only if your reasoning is clearly shown and easy to trace.
- If you use a theorem, formula, or table value, make it clear.
- When appropriate, carry quantities symbolically and substitute numerical values at the end.
- If necessary, round numerical answers to three decimal places.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Total	

Problem 1 (30 points total). Warm-up

(a) (6 points) Let A and B be events such that

$$P(A) = \frac{2}{5}, \qquad P(B) = \frac{1}{2}, \qquad P(A \cap B) = \frac{1}{5}.$$

Compute $P(A \cup B)$ and $P(A \mid B)$. Are A and B independent? Briefly justify your answer.

(b) (6 points) Let X have PMF

$$P(X = 0) = \frac{1}{4}, \qquad P(X = 1) = \frac{1}{2}, \qquad P(X = 2) = \frac{1}{4}.$$

Let $Y = (X - 1)^2$. Find the PMF of Y, and compute $\mathbb{E}[Y]$.

(c) (6 points) Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $P(X \ge 1/2)$ and $\mathbb{E}[X]$.

(d) (6 points) There are 8 students.

(i) How many ways are there to choose a president, a vice president, and a treasurer?

(ii) How many ways are there to choose a 3-person committee?

(e) (6 points) Suppose

$$X \sim N(70, 10^2).$$

Express $P(X > 85)$ in terms of the standard normal CDF Φ, and give a numerical approximation using the normal table.

Problem 2 (30 points total). Conditioning, Bayes' rule, and expectation

A hidden state H can take values A, B, C, with prior probabilities

$$P(H = A) = \frac{1}{4}, \qquad P(H = B) = \frac{1}{2}, \qquad P(H = C) = \frac{1}{4}.$$

You may think of A, B, C as three possible states of the world, such as strong, neutral, and weak market conditions. An event E may be observed, where the conditional probabilities of observing E are

$$P(E \mid H = A) = 0.8, \qquad P(E \mid H = B) = 0.3, \qquad P(E \mid H = C) = 0.1.$$

Note: For parts **(b)** and **(c)**, you may want to keep $P(E)$ symbolically in intermediate steps and substitute its value at the end.

(a) (10 points) Compute $P(E)$.

(b) (10 points) Compute the posterior probabilities

$$P(H = A \mid E), \qquad P(H = B \mid E), \qquad P(H = C \mid E).$$

(c) (10 points) Suppose that a payoff R depends only on the hidden state:

$$R = \begin{cases} 12, & H = A, \\ 4, & H = B, \\ -4, & H = C. \end{cases}$$

Compute $\mathbb{E}[R]$ and $\mathbb{E}[R \mid E]$.

Problem 3 (30 points total). Joint PDF and derived distribution

Suppose a task is completed in two stages. Let

$$X = \text{total completion time}, \qquad Y = \text{completion time of the first stage.}$$

Assume $0 \le Y \le X \le 1$, and suppose (X, Y) is jointly continuous with joint density

$$f_{X,Y}(x,y) = \begin{cases} c, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note: For parts **(b)** and **(c)**, you may want to carry the constant c symbolically in intermediate steps and substitute its value at the end.

(a) **(10 points)** Sketch the support of (X, Y), find the value of c, and determine whether X and Y are independent with a brief justification.

(b) **(10 points)** Find the marginal PDF $f_X(x)$. Then, for $0 < x \le 1$, find $f_{Y|X}(y \mid x)$ and compute

$$\mathbb{E}[Y \mid X = x].$$

(c) (10 points) Let

$$Z = X - Y,$$

the completion time of the second stage. Find the PDF of Z.

Problem 4 (30 points total). Dependence, total variance, and random sums

(a) **(10 points)** Suppose

$$\mathrm{Var}(X) = 9, \qquad \mathrm{Var}(Y) = 4, \qquad \mathrm{Cov}(X, Y) = -3.$$

Compute the correlation coefficient $\rho(X, Y)$, and compute

$$\mathrm{Var}(2X + Y).$$

(b) **(10 points)** Let $Z \sim \mathrm{Bernoulli}(1/4)$. Suppose

$$\mathbb{E}[X \mid Z = 0] = 0, \qquad \mathrm{Var}(X \mid Z = 0) = 1,$$
$$\mathbb{E}[X \mid Z = 1] = 4, \qquad \mathrm{Var}(X \mid Z = 1) = 9.$$

Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

(c) (10 points) Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli(0.2), and let $N \sim \text{Binomial}(10, 1/2)$ be independent of the sequence $X_1, X_2, \ldots$. Define

$$S = \sum_{i=1}^{N} X_i.$$

Compute $\mathbb{E}[S]$ and $\text{Var}(S)$. Then use MGFs to identify the distribution of S.

Problem 5 (30 points total). Deviation inequalities and the law of large numbers

(a) **(10 points)** Let $X \sim$ Exponential(1/2), where 1/2 is the rate parameter, i.e.,

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Compute $P(X \geq 6)$. Then use Markov's and Chebyshev's inequality to give two upper bounds.
(*Hint:* You may use $\mathbb{E}[X] = 2$ and $\mathrm{Var}(X) = 4$.)

(b) **(10 points)** Let $X_1, \ldots, X_n$ be i.i.d. random variables with

$$\mathbb{E}[X_i] = \mu, \qquad \mathrm{Var}(X_i) = 9.$$

Let

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Using Chebyshev's inequality, find a sufficient condition on the integer n to guarantee

$$P(|\bar{X}_n - \mu| \geq 0.2) \leq 0.05.$$

(c) (10 points) Consider two sequences of random variables:

$$Y_n = \begin{cases} n^2, & \text{with probability } 1/n, \\ 0, & \text{with probability } 1 - 1/n, \end{cases} \qquad Z_n = \begin{cases} 1, & \text{with probability } 1/2, \\ 0, & \text{with probability } 1/2. \end{cases}$$

For each sequence, determine whether it converges in probability to 0. Also determine whether its expectation converges, and if so, find the limit.

Problem 6 (30 points total). Central limit theorem and normal approximation

(a) (10 points) Let $X_1, \dots, X_n$ be i.i.d. with

$$\mathbb{E}[X_i] = \mu, \qquad \mathrm{Var}(X_i) = \sigma^2 < \infty, \qquad \sigma > 0.$$

Write the CLT approximation for

$$P(S_n \le c), \qquad \text{where} \qquad S_n = X_1 + \cdots + X_n,$$

expressing your answer in terms of the standard normal CDF Φ. Then find an approximate value of c, in terms of μ, σ, and n, such that

$$P(S_n \le c) \approx 0.95.$$

(b) (10 points) Suppose $X_1, \dots, X_{100}$ are i.i.d. Exponential(2), where 2 is the rate parameter. Let

$$\bar{X}_{100} = \frac{1}{100} \sum_{i=1}^{100} X_i.$$

Use the CLT to approximate

$$P(0.45 \le \bar{X}_{100} \le 0.55).$$

(c) **(10 points)** Let $S \sim \text{Binomial}(200, 0.5)$. Use the normal approximation with continuity correction to approximate

$$P(90 \leq S \leq 110).$$

Express your answer using Φ, and give a numerical approximation using the normal table.

Standard Normal Table

Entries give $\Phi(z) = P(Z \leq z)$ for $Z \sim N(0,1)$. For negative values, use $\Phi(-z) = 1 - \Phi(z)$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990