STA 131A Introduction to Probability Theory

Practice Final Exam – Version B

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Name: ____________________ **Student ID:** ____________________

Instructions. This practice final exam is **closed-book**, except for the permitted note sheets described below. You may bring a pen or pencil, three letter-sized sheets of *hand-written* notes (both sides), and a simple *non-graphing* calculator, but nothing else. You have 2 hours to complete the exam. The **total score is 180 points**. A standard normal table is included at the end of the exam.

- Make sure to clearly write your name and student ID above.
- Present answers succinctly, but include all relevant steps for full credit.
- If you use a theorem, formula, or table value, make it clear.
- Partial credit is possible only if your reasoning is clearly shown and traceable by the grader.
- If necessary, round numerical answers to three decimal places.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Total	

Problem 1 (30 points in total). Warm-up

(a) **(5 points)** Let A and B be events such that

$$P(A) = \frac{3}{5}, \qquad P(B) = \frac{1}{2}, \qquad P(A \cup B) = \frac{4}{5}.$$

Compute $P(A \cap B)$ and $P(A \mid B)$. Are A and B independent? Briefly justify your answer.

(b) **(5 points)** Let X have PMF

$$P(X = -2) = \frac{1}{6}, \qquad P(X = 0) = \frac{1}{3}, \qquad P(X = 1) = \frac{1}{2}.$$

Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

(c) **(5 points)** A password consists of 4 distinct digits chosen from $0, 1, \ldots, 9$. The first digit cannot be 0. How many such passwords are possible?

(d) (5 points) Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 2(1-x), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $P(X \le 1/2)$ and $\mathbb{E}[X]$.

(e) (5 points) Suppose

$$\text{Var}(X) = 4, \qquad \text{Var}(Y) = 9, \qquad \text{Cov}(X,Y) = 3.$$

Compute $\rho(X,Y)$ and $\text{Var}(2X - Y)$.

(f) (5 points) Suppose

$$X \sim N(100, 15^2).$$

Express $P(85 \le X \le 130)$ in terms of the standard normal CDF Φ, and give a numerical approximation using the normal table.

Problem 2 (30 points in total). Hidden state, posterior weights, and prediction

A hidden state H is chosen uniformly from three possible states A, B, C:

$$P(H = A) = P(H = B) = P(H = C) = \frac{1}{3}.$$

Conditional on H, three independent alerts are generated, each with success probability

$$P(\text{success} \mid H = A) = 0.8, \qquad P(\text{success} \mid H = B) = 0.5, \qquad P(\text{success} \mid H = C) = 0.2.$$

Let

$$E = \{\text{exactly two of the three alerts are successes}\}.$$

(a) (10 points) Compute $P(E \mid H = A)$, $P(E \mid H = B)$, $P(E \mid H = C)$, and $P(E)$.

(b) (10 points) Define the unnormalized posterior weights

$$w_h = P(E \mid H = h)P(H = h), \qquad h \in \{A, B, C\}.$$

Compute w_A, w_B, w_C, and use them to rank the hidden states from most likely to least likely after observing E.

(c) **(10 points)** After observing E, two additional alerts are generated under the same hidden state H, independently with the corresponding success probability. Compute the posterior predictive probability that exactly one of the two additional alerts is a success:

$$P(\text{exactly one success among the next two alerts} \mid E).$$

Hint: You may use the unnormalized weights from part **(b)** and normalize only at the end.

Problem 3 (30 points in total). Conditional discrete model

Let G be a discrete random variable with

$$P(G = 0) = \frac{1}{2}, \qquad P(G = 1) = \frac{1}{3}, \qquad P(G = 2) = \frac{1}{6}.$$

Conditional on $G = g$, the conditional PMF of X is given by

$p_{X\mid G}(x \mid g)$	$x = 0$	$x = 1$	$x = 2$	$x = 4$
$g = 0$	1/2	0	1/2	0
$g = 1$	0	1	0	0
$g = 2$	1/2	0	0	1/2

(a) **(10 points)** Find the marginal PMF of X.

(b) **(10 points)** Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. You may use the law of total expectation and the law of total variance.

(c) **(10 points)** Compute $\mathrm{Cov}(X, G)$ and $\rho(X, G)$. Are X and G independent? Briefly justify your answer.

Problem 4 (30 points in total). Joint PDF and derived distribution

An operation has two nonnegative components X and Y. Suppose (X, Y) is uniformly distributed over the region

$$R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1 + x\}.$$

That is,

$$f_{X,Y}(x, y) = \begin{cases} c, & (x, y) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$T = X + Y.$$

(a) (10 points) Sketch R, and find the value of c.

(b) (10 points) Find the marginal PDF $f_Y(y)$, giving a piecewise expression if needed. Then find $f_{X|Y}(x \mid y)$ and compute $\mathbb{E}[X \mid Y = y]$ for $0 < y < 2$.

(c) (10 points) Find the CDF and PDF of $T = X + Y$.

Problem 5 (30 points in total). Tail bounds, convergence, and CLT

(a) (10 points) Let X have PDF

$$f_X(x) = \begin{cases} xe^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Compute $P(X \geq 5)$. Then use Markov's and Chebyshev's inequalities to give two upper bounds for $P(X \geq 5)$.

Hint: You may use $\mathbb{E}[X] = 2$, $\mathrm{Var}(X) = 2$, and

$$\int_a^\infty xe^{-x}\, dx = (a+1)e^{-a}.$$

(b) (10 points) Let $U \sim \mathrm{Uniform}(0,1)$, and define

$$Y_n = nU^n.$$

Determine whether $Y_n \xrightarrow{P} 0$. Also compute $\mathbb{E}[Y_n]$ and determine whether $\mathbb{E}[Y_n] \to 0$.

Hint: For any fixed $a > 0$, $(a/n)^{1/n} \to 1$.

(c) (10 points) Let $U_1, \ldots, U_{500}$ be i.i.d. Uniform$(0, 1)$, and define

$$A_{500} = \frac{1}{500} \sum_{i=1}^{500} U_i^2.$$

Use the CLT to approximate

$$P\left(\left| A_{500} - \frac{1}{3} \right| \leq 0.02 \right).$$

Express your answer using Φ, then give a numerical approximation.

Hint: If $U \sim$ Uniform$(0, 1)$, then $\mathbb{E}[U^2] = 1/3$ and $\text{Var}(U^2) = 4/45$.

Problem 6 (30 points in total). Inverse problem and statistical estimation

Consider an unknown signal Θ and a noisy measurement

$$Y = \Theta + \varepsilon,$$

where

$$\Theta \sim \text{Uniform}(0, 2), \qquad \varepsilon \sim \text{Uniform}(-1, 1),$$

and Θ and ε are independent.

(a) (10 points) Compute $\mathbb{E}[Y]$ and $\text{Var}(Y)$.

(b) (10 points) Suppose we observe the event

$$B = \{0 \le Y \le 1/2\}.$$

Find the posterior density $f_{\Theta|B}(\theta)$, for $0 \le \theta \le 2$. Then compute $\mathbb{E}[\Theta \mid B]$.

Hint: First compute $P(B \mid \Theta = \theta)$ as a function of θ.

(c) (10 points) Suppose now that $\Theta = \theta$ is fixed, and we collect independent measurements

$$Y_i = \theta + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where the ε_i's are i.i.d. Uniform$(-1, 1)$. Let

$$\bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

Find two sample-size recommendations:

(i) a sufficient integer n using Chebyshev's inequality;

(ii) an approximate integer n using the CLT,

so that

$$P(|\bar{Y}_n - \theta| \leq 0.1 \mid \Theta = \theta) \geq 0.95.$$

Standard Normal Table

Entries give $\Phi(z) = P(Z \leq z)$ for $Z \sim N(0, 1)$. For negative values, use $\Phi(-z) = 1 - \Phi(z)$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990