

# STA 131A Introduction to Probability Theory

## Midterm exam 1

Instructor: Dogyoon Song

Name: \_\_\_\_\_ Student ID: \_\_\_\_\_

**Instructions:** This midterm exam is **closed-book**, except for the permitted note sheet described below. You may bring a pen or pencil, one letter-sized sheet of *hand-written* notes (both sides), and a *non-graphing* calculator. No other materials (e.g., textbooks) are allowed. You have 50 minutes to complete all problems. The **total score is 120 points**, with *up to 10 bonus points available*. Once you receive this exam, please confirm that you have all 8 pages.

- Make sure to clearly write your name and student ID above.
- Present answers succinctly, but include all relevant steps for full credit. Partial credit is possible only if your reasoning is clearly shown and traceable by the grader.
- If necessary, round all numerical answers to three decimal places; you may leave fractions such as  $8/55$  unevaluated.
- Bonus problems can be more challenging; consider attempting them after you finish the main problems.

Problem	Score
Problem 1	
Problem 2	
Problem 3	
Problem 4	
<b>Total</b>	

**Problem 1 (30 points in total). Warm-up**

(a) (6 points) Let  $A$  and  $B$  be events such that

$$P(A) = \frac{2}{5}, \quad P(B) = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{5}.$$

Compute  $P(A \cup B)$ ,  $P(A^c \cap B)$ , and  $P(A | B)$ . Are  $A$  and  $B$  independent?

(b) (6 points) Let  $X$  be a discrete random variable with PMF

$$p_X(x) = \begin{cases} \frac{1}{4}, & x = -1, \\ \frac{1}{2}, & x = 0, \\ \frac{1}{4}, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $Y = X^2$ . Find the PMF of  $Y$ , and compute  $\mathbb{E}[Y]$ .

(c) (6 points) There are 10 students.

(i) How many ways are there to choose a president, a vice president and a treasurer, assuming no student can hold more than one position?

(ii) How many ways are there to choose a 2-person committee?

(d) **(6 points)** A student answers 5 true/false questions independently. Each answer is correct with probability  $3/4$ . What is the probability that the student answers exactly 4 questions correctly?

(e) **(6 points)** Independent Bernoulli trials are repeated, each with success probability  $1/4$ . Let  $N$  be the trial number on which the first success occurs. Compute  $P(N = 3)$ .

**Problem 2 (30 points in total). Quality control**

A factory produces items using three machines. Let  $M_i$  denote the event that a randomly selected item was produced by machine  $i \in \{1, 2, 3\}$ . The source probabilities are

$$P(M_1) = \frac{1}{2}, \quad P(M_2) = \frac{1}{3}, \quad P(M_3) = \frac{1}{6}.$$

Let  $D$  be the event that an item is defective. The defect rates for each machine are

$$P(D | M_1) = 0.02, \quad P(D | M_2) = 0.05, \quad P(D | M_3) = 0.08.$$

(a) (10 points) Compute  $P(D)$ .

(b) (10 points) Given that an item is defective, what is the probability that it came from machine 3?

(c) (10 points) Consider a diagnostic test applied to a randomly selected item, designed to flag defective items. Let  $T$  be the event that the test is positive. Suppose

$$P(T | D) = 0.90, \quad P(T | D^c) = 0.05.$$

Compute  $P(D | T)$ .

**Problem 3 (30 points in total + 5 bonus points). Sampling from an urn**

An urn contains 4 red balls and 6 blue balls. Three balls are selected uniformly at random without replacement, meaning that every 3-ball subset of the 10 individual balls is equally likely to be chosen. Let

$X =$  the number of red balls selected.

(a) (10 points) Find the PMF of  $X$ , and compute  $\mathbb{E}[X]$ .

(b) (10 points) Compute the following probabilities either by using the PMF above or by direct counting:

$P(\text{at least one red ball is selected})$       and       $P(\text{all three selected balls have the same color})$ .

- (c) **(10 points)** Now suppose that we draw three balls sequentially one by one *with replacement*, so that after each draw the selected ball is returned to the urn before the next draw. Let

$Z$  = the number of red balls selected in the three draws.

Find the PMF of  $Z$ , and compute  $\text{Var}(Z)$ .

- (d\*) **(5 bonus points)** Suppose that we draw balls *with replacement*. The process is repeated until, for the first time, we see two consecutive reds or two consecutive blues. Let

$W$  = the number of draws until termination.

Compute  $P(W = 1)$ ,  $P(W = 2)$ , and  $P(W = 3)$ . Then find a formula for the PMF of  $W$ , i.e., for  $P(W = w)$  for all possible values of  $w$ .

**Problem 4 (30 points in total + 5 bonus points). Joint and conditional PMFs**

The joint PMF of two discrete random variables  $X$  and  $Y$  is

$p_{X,Y}(x,y)$	$y = 0$	$y = 1$	$y = 2$
$x = 0$	1/8	1/8	1/4
$x = 1$	1/4	1/8	1/8

and  $p_{X,Y}(x,y) = 0$  for all other pairs.

(a) (10 points) Find the marginal PMFs  $p_X$  and  $p_Y$ . Then compute  $P(X < Y)$ .

(b) (10 points) For each  $x \in \{0,1\}$ , compute the conditional PMF  $p_{Y|X}(y | x)$  and the conditional expectation  $\mathbb{E}[Y | X = x]$ .

- (c) **(10 points)** Use the law of total expectation to compute  $\mathbb{E}[Y]$  using  $\mathbb{E}[Y \mid X = x]$ . Then compute  $\mathbb{E}[XY]$ , and determine whether  $X$  and  $Y$  are independent. Justify your answer.

- (d\*) **(5 bonus points)** Among all joint PMFs supported on  $\{0, 1\} \times \{0, 1, 2\}$  with marginals

$$p_X(0) = p_X(1) = \frac{1}{2}, \quad p_Y(0) = \frac{3}{8}, \quad p_Y(1) = \frac{1}{4}, \quad p_Y(2) = \frac{3}{8},$$

find the largest possible value of  $\text{Var}(X + Y)$  among such joint PMFs, and write a joint PMF table for  $(X, Y)$  that attains it.