STA 131A Introduction to Probability Theory

(Mock Exam for Midterm 1 - Version A)

Instructor: Dogyoon Song

Name: ______________________ **Student ID:** __________________

Instructions. This practice midterm is designed to resemble a 50-minute in-class exam. However, the actual Midterm 1 may differ in content or style from this practice exam. Assume you may use one *hand-written*, two-sided, letter-sized cheat sheet and a simple non-graphing calculator. The **total score is 120 points**, with **up to 10 bonus points** available.

- Make sure to clearly write your name and ID above.
- Present your answers clearly and show enough work to justify your conclusions for full credit.
- Partial credit is possible only if your reasoning is clearly shown and easy to trace.
- If you use a theorem or formula, make it clear.
- Bonus subproblems are optional and may be more exploratory.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Total	

Problem 1 (30 points total).

Two fair six-sided dice are rolled. Each of the 36 ordered outcomes is equally likely.

(a) (10 points) Let

$$A = \{\text{at least one die shows } 6\}, \qquad B = \{\text{the two dice show different numbers}\}.$$

Compute $P(A \mid B)$.

(b) (10 points) Let

$$C = \{\text{the sum is } 7\}, \qquad D = \{\text{the first die shows } 4\}.$$

Are C and D independent? Justify your answer.

(c) (10 points) Let

$$E = \{\text{the first die shows } 3\}, \qquad F = \{\text{the second die shows } 5\}, \qquad G = \{\text{the sum is } 8\}.$$

Are E and F conditionally independent given G? Justify your answer.

Problem 2 (30 points total + 5 bonus points).

Players A and B play independent games. In each game, player A wins with probability p, where $0 < p < 1$, and player B wins with probability $1 - p$.

They play a best-of-3 match, meaning the match ends as soon as one player has won two games.

Let L be the number of games played, and let W be the event that player A wins the match.

(a) (8 points) List all possible match-ending sequences of winners. Hence compute

$$P(L = 2) \qquad \text{and} \qquad P(L = 3).$$

(b) (10 points) Compute $P(W)$, the probability that player A wins the best-of-3 match.

(c) (12 points) Show that

$$P(W) > p \quad \text{if } p > \frac{1}{2}, \qquad P(W) = p \quad \text{if } p = \frac{1}{2}, \qquad P(W) < p \quad \text{if } p < \frac{1}{2}.$$

In other words, show that a best-of-3 match amplifies the advantage of the stronger player.

(d*) (5 bonus points) Compute $\mathbb{E}[L]$, the expected number of games played in the best-of-3 match, as a function of p.

Problem 3 (25 points total + 5 bonus points).

Let X be a discrete random variable with

$$P(X=-1)=\frac{1}{4}, \qquad P(X=0)=\frac{1}{2}, \qquad P(X=2)=\frac{1}{4}.$$

(a) (8 points) Compute $\mathbb{E}[X]$, $\mathbb{E}[X^2]$, and $\mathrm{Var}(X)$.

(b) (7 points) Let $Y=X^2$. Find the PMF of Y, and compute $\mathbb{E}[Y]$.

(c) (10 points) Let $a\in\mathbb{R}$. Prove that

$$\mathbb{E}[(X-a)^2]=\mathrm{Var}(X)+\big(\mathbb{E}[X]-a\big)^2.$$

Hence determine the value of a that minimizes $\mathbb{E}[(X-a)^2]$.

(d*) (5 bonus points) Independently and uniformly, n balls are thrown into n bins. Let Z be the number of empty bins. Compute $\mathbb{E}[Z]$.

(*Hint:* Use indicator random variables.)

Problem 4 (35 points total).

A fair three-sided die with faces $1, 2, 3$ is rolled once. Let X denote the die outcome.

Then a fair coin is tossed X times, and Y denotes the number of Heads observed.

(a) **(10 points)** Determine the joint PMF $p_{X,Y}(x, y)$ for all pairs (x, y) with positive probability, and present your answer in a table.

(b) **(10 points)** Compute $p_Y(1)$, then find the conditional PMF $p_{X|Y}(x \mid 1)$. Finally, compute $\mathbb{E}[X \mid Y = 1]$.

(c) **(10 points)** Compute $\mathbb{E}[Y \mid X = x]$ for each $x \in \{1, 2, 3\}$. Then use the law of total expectation to compute $\mathbb{E}[Y]$.

(d) **(5 points)** Are X and Y independent random variables? Justify your answer carefully.