STA 131A Introduction to Probability Theory

(Practice Midterm 1 - Version A Solution)

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Problem 1

(a) B has $36 - 6 = 30$ outcomes. $A \cap B$ consists of the 10 outcomes with exactly one 6:

$$(6, 1), \ldots, (6, 5), (1, 6), \ldots, (5, 6).$$

Hence

$$P(A \mid B) = \frac{10/36}{30/36} = \frac{1}{3}.$$

(b)

$$P(C) = \frac{6}{36} = \frac{1}{6}, \qquad P(D) = \frac{6}{36} = \frac{1}{6}, \qquad P(C \cap D) = P((4, 3)) = \frac{1}{36}.$$

Since

$$P(C \cap D) = \frac{1}{36} = P(C)P(D),$$

the events C and D are independent.

(c) Given G, the possible outcomes are

$$(2, 6), (3, 5), (4, 4), (5, 3), (6, 2),$$

each with conditional probability $1/5$. Therefore

$$P(E \mid G) = \frac{1}{5}, \qquad P(F \mid G) = \frac{1}{5}, \qquad P(E \cap F \mid G) = \frac{1}{5}.$$

Since

$$\frac{1}{5} \neq \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25},$$

E and F are not conditionally independent given G.

Problem 2

(a) The possible match-ending sequences are

$$AA, \; BB, \; ABA, \; ABB, \; BAA, \; BAB.$$

Hence

$$P(L = 2) = P(AA) + P(BB) = p^2 + (1 - p)^2,$$

$$P(L = 3) = P(ABA) + P(ABB) + P(BAA) + P(BAB) = 2p(1 - p).$$

(b) Player A wins via the sequences AA, ABA, BAA. Thus

$$P(W) = p^2 + p(1-p)p + (1-p)pp = p^2 + 2p^2(1-p) = p^2(3-2p).$$

(c) Compute

$$P(W) - p = p^2(3-2p) - p = p\big(3p - 2p^2 - 1\big) = p(1-p)(2p-1).$$

Since $p(1-p) > 0$ for $0 < p < 1$, the sign of $P(W) - p$ is the sign of $2p - 1$. Therefore

$$P(W) > p \text{ if } p > \tfrac{1}{2}, \qquad P(W) = p \text{ if } p = \tfrac{1}{2}, \qquad P(W) < p \text{ if } p < \tfrac{1}{2}.$$

(d*)

$$\mathbb{E}[L] = 2P(L=2) + 3P(L=3) = 2\big(p^2 + (1-p)^2\big) + 3\big(2p(1-p)\big).$$

Simplifying,

$$\mathbb{E}[L] = 2 + 2p(1-p).$$

Problem 3

(a)

$$\mathbb{E}[X] = (-1)\cdot\frac{1}{4} + 0\cdot\frac{1}{2} + 2\cdot\frac{1}{4} = \frac{1}{4}.$$

$$\mathbb{E}[X^2] = 1\cdot\frac{1}{4} + 0\cdot\frac{1}{2} + 4\cdot\frac{1}{4} = \frac{5}{4}.$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = \frac{5}{4} - \left(\frac{1}{4}\right)^2 = \frac{19}{16}.$$

(b) $Y = X^2$ takes values $0, 1, 4$, with

$$P(Y=0) = \frac{1}{2}, \qquad P(Y=1) = \frac{1}{4}, \qquad P(Y=4) = \frac{1}{4}.$$

Therefore

$$\mathbb{E}[Y] = 0\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 4\cdot\frac{1}{4} = \frac{5}{4}.$$

(c) Expand:

$$\mathbb{E}[(X-a)^2] = \mathbb{E}[X^2 - 2aX + a^2] = \mathbb{E}[X^2] - 2a\mathbb{E}[X] + a^2.$$

Since

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2,$$

we obtain

$$\mathbb{E}[(X-a)^2] = \mathrm{Var}(X) + \big(\mathbb{E}[X]\big)^2 - 2a\mathbb{E}[X] + a^2 = \mathrm{Var}(X) + (\mathbb{E}[X] - a)^2.$$

Because $(\mathbb{E}[X] - a)^2 \geq 0$, the minimum occurs at

$$a = \mathbb{E}[X] = \frac{1}{4}.$$

(d*) For $i = 1, \ldots, n$, let

$$I_i = \begin{cases} 1, & \text{if bin } i \text{ is empty,} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$Z = I_1 + \cdots + I_n.$$

A given bin is empty if all n balls avoid it, which has probability

$$P(I_i = 1) = \left(1 - \frac{1}{n}\right)^n.$$

Therefore,

$$\mathbb{E}[Z] = \sum_{i=1}^{n} \mathbb{E}[I_i] = n\left(1 - \frac{1}{n}\right)^n.$$

Problem 4

(a) Since $P(X = x) = 1/3$ for $x = 1, 2, 3$, and $Y \mid X = x \sim \text{Binomial}(x, 1/2)$, the joint PMF table is

$p_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$y=3$
$x=1$	1/6	1/6	0	0
$x=2$	1/12	1/6	1/12	0
$x=3$	1/24	1/8	1/8	1/24

(b)

$$p_Y(1) = \frac{1}{6} + \frac{1}{6} + \frac{1}{8} = \frac{11}{24}.$$

Hence

$$P(X = 1 \mid Y = 1) = \frac{1/6}{11/24} = \frac{4}{11}, \qquad P(X = 2 \mid Y = 1) = \frac{1/6}{11/24} = \frac{4}{11}, \qquad P(X = 3 \mid Y = 1) = \frac{1/8}{11/24} = \frac{3}{11}.$$

Therefore

$$\mathbb{E}[X \mid Y = 1] = 1 \cdot \frac{4}{11} + 2 \cdot \frac{4}{11} + 3 \cdot \frac{3}{11} = \frac{21}{11}.$$

(c) Since $Y \mid X = x \sim \text{Binomial}(x, 1/2)$,

$$\mathbb{E}[Y \mid X = x] = \frac{x}{2}, \qquad x = 1, 2, 3.$$

Thus

$$\mathbb{E}[Y] = \sum_{x=1}^{3} \mathbb{E}[Y \mid X = x] P(X = x) = \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + \frac{3}{2} \cdot \frac{1}{3} = 1.$$

(d) No. For example,

$$p_{X,Y}(1,1) = \frac{1}{6},$$

whereas

$$p_X(1) p_Y(1) = \frac{1}{3} \cdot \frac{11}{24} = \frac{11}{72}.$$

Since

$$\frac{1}{6} = \frac{12}{72} \neq \frac{11}{72},$$

X and Y are not independent.