STA 131A Introduction to Probability Theory

(Mock Exam for Midterm 1 - Version B)

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Name: ______________________ **Student ID:** __________________

Instructions. This practice midterm is designed to resemble a 50-minute in-class exam. However, the actual Midterm 1 may differ in content or style from this practice exam. Assume you may use one *hand-written*, two-sided, letter-sized cheat sheet and a simple non-graphing calculator. The **total score is 120 points**, with **up to 10 bonus points** available.

- Make sure to clearly write your name and ID above.
- Present your answers clearly and show enough work to justify your conclusions for full credit.
- Partial credit is possible only if your reasoning is clearly shown and easy to trace.
- If you use a theorem or formula, make it clear.
- Bonus subproblems are optional and may be more exploratory.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Total	

Problem 1 (30 points total + 5 bonus points).

A box is chosen at random:

- Box A is chosen with probability 1/2, and each draw from Box A is red with probability 3/4.
- Box B is chosen with probability 1/2, and each draw from Box B is red with probability 1/4.

After the box is chosen, two balls are drawn *with replacement* from that box.

Let

$$C = \{\text{Box A is chosen}\}, \qquad R_1 = \{\text{the first draw is red}\}, \qquad R_2 = \{\text{the second draw is red}\}.$$

(a) (10 points) Compute $P(C \mid R_1)$.

(b) (10 points) Are R_1 and R_2 independent? Justify your answer.

(c) (10 points) Are R_1 and R_2 conditionally independent given C? Justify your answer.

(d*) (5 bonus points) If both draws are red, what is the posterior probability that Box A was chosen? That is, compute

$$P(C \mid R_1 \cap R_2).$$

Problem 2 (30 points total).

A box contains 4 red balls, 3 blue balls, and 3 green balls. Three balls are drawn *without replacement.*

(a) (10 points) How many unordered 3-ball samples are possible?

(b) (10 points) What is the probability that the sample contains exactly one red ball, one blue ball, and one green ball?

(c) (10 points) Now the whole 3-ball experiment is repeated independently: after each trial, all balls are returned to the box and mixed before drawing again. Let T be the number of trials until the first time the sample contains one red ball, one blue ball, and one green ball.

Identify the distribution of T by name and parameter(s), and compute $P(T = 2)$.

Problem 3 (25 points total).

Let X be the outcome of a fair six-sided die roll.

(a) (10 points) Compute $\mathbb{E}[X]$, $\mathbb{E}[X^2]$, and $\mathrm{Var}(X)$.

(b) (5 points) Let $Y = 2X - 1$. Compute $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

(c) (10 points) Let $\mathbf{1}_A$ denote the indicator random variable of an event A, i.e.

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Prove that

$$\mathbb{E}[\mathbf{1}_A] = P(A) \qquad \text{and} \qquad \mathrm{Var}(\mathbf{1}_A) = P(A)\big(1 - P(A)\big).$$

Problem 4 (35 points total + 5 bonus points).

Suppose (X, Y) has joint PMF

$p_{X,Y}(x,y)$	$y=0$	$y=1$
$x=0$	$1/8$	$1/8$
$x=1$	$1/4$	$1/8$
$x=2$	0	$3/8$

and $p_{X,Y}(x,y) = 0$ for all other pairs (x, y).

(a) (10 points) Find the marginal PMFs p_X and p_Y. Then compute $P(X > Y)$.

(b) (10 points) Find the conditional PMF $p_{X|Y}(x \mid 1)$, and compute $\mathbb{E}[X \mid Y = 1]$.

(c) (10 points) Compute $\mathbb{E}[X \mid Y = 0]$, and then use the law of total expectation to find $\mathbb{E}[X]$.

(d) (5 points) Determine whether X and Y are independent random variables. Justify your answer.

(e*) (5 bonus points) Let $Z = X + Y$. Find the PMF of Z.