STA 131A Introduction to Probability Theory

(Practice Midterm 1 - Version B Solution)

Instructor: Dogyoon Song

Problem 1

(a)

$$P(R_1) = \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2}.$$

Hence

$$P(C \mid R_1) = \frac{P(R_1 \mid C)P(C)}{P(R_1)} = \frac{(3/4)(1/2)}{1/2} = \frac{3}{4}.$$

(b)

$$P(R_1) = P(R_2) = \frac{1}{2}.$$

Also,

$$P(R_1 \cap R_2) = \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{4}\right)^2 = \frac{9}{32} + \frac{1}{32} = \frac{5}{16}.$$

Since

$$\frac{5}{16} \neq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$

R_1 and R_2 are not independent.

(c) Given C, both draws come from Box A, with replacement, so the draws are independent and

$$P(R_1 \mid C) = P(R_2 \mid C) = \frac{3}{4}.$$

Also,

$$P(R_1 \cap R_2 \mid C) = \left(\frac{3}{4}\right)^2 = \frac{9}{16} = P(R_1 \mid C)P(R_2 \mid C).$$

Therefore R_1 and R_2 are conditionally independent given C.

(d*)

$$P(R_1 \cap R_2 \mid C) = \left(\frac{3}{4}\right)^2 = \frac{9}{16}, \qquad P(R_1 \cap R_2 \mid C^c) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

Hence

$$P(R_1 \cap R_2) = \frac{1}{2} \cdot \frac{9}{16} + \frac{1}{2} \cdot \frac{1}{16} = \frac{10}{32} = \frac{5}{16}.$$

Therefore

$$P(C \mid R_1 \cap R_2) = \frac{(9/16)(1/2)}{5/16} = \frac{9}{10}.$$

Problem 2

(a) The number of unordered 3-ball samples is

$$\binom{10}{3} = 120.$$

(b) To get exactly one red, one blue, and one green ball, the number of favorable samples is

$$\binom{4}{1}\binom{3}{1}\binom{3}{1} = 4 \cdot 3 \cdot 3 = 36.$$

Hence

$$P(\text{one red, one blue, one green}) = \frac{36}{120} = \frac{3}{10}.$$

(c) Each trial is independent, and success means "the sample contains one red, one blue, and one green," which has probability $p = 3/10$. Thus

$$T \sim \text{Geometric}(3/10).$$

Therefore

$$P(T = 2) = \left(1 - \frac{3}{10}\right)\frac{3}{10} = \frac{7}{10} \cdot \frac{3}{10} = \frac{21}{100}.$$

Problem 3

(a)

$$\mathbb{E}[X] = \frac{1+2+3+4+5+6}{6} = \frac{7}{2},$$
$$\mathbb{E}[X^2] = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \frac{91}{6},$$
$$\text{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

(b) Since $Y = 2X - 1$,

$$\mathbb{E}[Y] = 2\mathbb{E}[X] - 1 = 2 \cdot \frac{7}{2} - 1 = 6,$$

and

$$\text{Var}(Y) = \text{Var}(2X - 1) = 4\text{Var}(X) = 4 \cdot \frac{35}{12} = \frac{35}{3}.$$

(c) Since $\mathbf{1}_A$ takes value 1 on A and 0 on A^c,

$$\mathbb{E}[\mathbf{1}_A] = 1 \cdot P(A) + 0 \cdot P(A^c) = P(A).$$

Also, $\mathbf{1}_A^2 = \mathbf{1}_A$, so

$$\text{Var}(\mathbf{1}_A) = \mathbb{E}[\mathbf{1}_A^2] - \big(\mathbb{E}[\mathbf{1}_A]\big)^2 = \mathbb{E}[\mathbf{1}_A] - P(A)^2 = P(A) - P(A)^2 = P(A)(1 - P(A)).$$

Problem 4

(a)

$$p_X(0) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}, \qquad p_X(1) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}, \qquad p_X(2) = 0 + \frac{3}{8} = \frac{3}{8}.$$

$$p_Y(0) = \frac{1}{8} + \frac{1}{4} + 0 = \frac{3}{8}, \qquad p_Y(1) = \frac{1}{8} + \frac{1}{8} + \frac{3}{8} = \frac{5}{8}.$$

Also,

$$P(X > Y) = P(1,0) + P(2,1) = \frac{1}{4} + \frac{3}{8} = \frac{5}{8}.$$

(b) Since $p_Y(1) = 5/8$,

$$P(X = 0 \mid Y = 1) = \frac{1/8}{5/8} = \frac{1}{5}, \qquad P(X = 1 \mid Y = 1) = \frac{1/8}{5/8} = \frac{1}{5}, \qquad P(X = 2 \mid Y = 1) = \frac{3/8}{5/8} = \frac{3}{5}.$$

Therefore

$$\mathbb{E}[X \mid Y = 1] = 0 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5} + 2 \cdot \frac{3}{5} = \frac{7}{5}.$$

(c) Since $p_Y(0) = 3/8$,

$$P(X = 0 \mid Y = 0) = \frac{1/8}{3/8} = \frac{1}{3}, \qquad P(X = 1 \mid Y = 0) = \frac{1/4}{3/8} = \frac{2}{3}, \qquad P(X = 2 \mid Y = 0) = 0.$$

Hence

$$\mathbb{E}[X \mid Y = 0] = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} + 2 \cdot 0 = \frac{2}{3}.$$

By the law of total expectation,

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X \mid Y = 0]P(Y = 0) + \mathbb{E}[X \mid Y = 1]P(Y = 1) \\ &= \frac{2}{3} \cdot \frac{3}{8} + \frac{7}{5} \cdot \frac{5}{8} = \frac{1}{4} + \frac{7}{8} = \frac{9}{8}. \end{aligned}$$

(d) X and Y are not independent because

$$p_{X,Y}(0,0) = \frac{1}{8}, \qquad p_X(0)p_Y(0) = \frac{1}{4} \cdot \frac{3}{8} = \frac{3}{32}.$$

Since

$$\frac{1}{8} \neq \frac{3}{32},$$

X and Y are not independent.

(e*) Let $Z = X + Y$. Then

$$\begin{aligned} P(Z = 0) &= P(0,0) = \frac{1}{8}, \\ P(Z = 1) &= P(0,1) + P(1,0) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}, \\ P(Z = 2) &= P(1,1) + P(2,0) = \frac{1}{8} + 0 = \frac{1}{8}, \\ P(Z = 3) &= P(2,1) = \frac{3}{8}. \end{aligned}$$

So the PMF of Z is

$$P(Z = 0) = \frac{1}{8}, \quad P(Z = 1) = \frac{3}{8}, \quad P(Z = 2) = \frac{1}{8}, \quad P(Z = 3) = \frac{3}{8}.$$