STA 131A Introduction to Probability Theory

Midterm exam 1 solution

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Problem 1. Warm-up

(a) We are given

$$P(A) = \frac{2}{5}, \qquad P(B) = \frac{1}{2}, \qquad P(A \cap B) = \frac{1}{5}.$$

Therefore,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) = \frac{2}{5} + \frac{1}{2} - \frac{1}{5} = \frac{7}{10}, \\ P(A^c \cap B) &= P(B) - P(A \cap B) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}, \\ P(A \mid B) &= \frac{P(A \cap B)}{P(B)} = \frac{1/5}{1/2} = \frac{2}{5}. \end{aligned}$$

Since

$$P(A)P(B) = \frac{2}{5} \cdot \frac{1}{2} = \frac{1}{5} = P(A \cap B),$$

the events A and B are independent.

(b) Since $Y = X^2$, we have

$$Y = 0 \quad \text{if } X = 0, \qquad Y = 1 \quad \text{if } X = -1 \text{ or } X = 1.$$

Therefore,

$$\begin{aligned} P(Y = 0) &= P(X = 0) = \frac{1}{2}, \\ P(Y = 1) &= P(X = -1) + P(X = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

Thus,

$$p_Y(y) = \begin{cases} \frac{1}{2}, & y = 0, \\ \frac{1}{2}, & y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\mathbb{E}[Y] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

(c)(i) The roles are distinct and order matters. There are 10 choices for president, then 9 choices for vice president, and then 8 choices for treasurer. Thus,

$$10 \cdot 9 \cdot 8 = 720.$$

(ii) A 2-person committee is unordered, so the number of choices is

$$\binom{10}{2} = \frac{10 \cdot 9}{2} = 45.$$

(d) Let Z be the number of correct answers. Then

$$Z \sim \text{Binomial}\left(5, \frac{3}{4}\right).$$

Therefore,

$$P(Z = 4) = \binom{5}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) = 5 \cdot \frac{81}{256} \cdot \frac{1}{4} = \frac{405}{1024} \approx 0.396.$$

(e) Since N is the trial number of the first success,

$$P(N = 3) = \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) = \frac{9}{64}.$$

Problem 2. Quality control

(a) By the law of total probability,

$$\begin{aligned} P(D) &= P(D \mid M_1)P(M_1) + P(D \mid M_2)P(M_2) + P(D \mid M_3)P(M_3) \\ &= 0.02 \cdot \frac{1}{2} + 0.05 \cdot \frac{1}{3} + 0.08 \cdot \frac{1}{6} \\ &= \frac{1}{100} + \frac{1}{60} + \frac{1}{75} \\ &= \frac{12}{300} = 0.04. \end{aligned}$$

(b) By Bayes' rule,

$$P(M_3 \mid D) = \frac{P(D \mid M_3)P(M_3)}{P(D)} = \frac{0.08 \cdot \frac{1}{6}}{0.04} = \frac{1/75}{1/25} = \frac{1}{3}.$$

(c) We use Bayes' rule again:

$$P(D \mid T) = \frac{P(T \mid D)P(D)}{P(T \mid D)P(D) + P(T \mid D^c)P(D^c)}.$$

Since $P(D) = 0.04$, we have $P(D^c) = 0.96$. Therefore,

$$P(D \mid T) = \frac{0.90 \cdot 0.04}{0.90 \cdot 0.04 + 0.05 \cdot 0.96} = \frac{0.036}{0.036 + 0.048} = \frac{0.036}{0.084} = \frac{3}{7} \approx 0.429.$$

Problem 3. Sampling from an urn

(a) There are 4 red balls and 6 blue balls, and 3 balls are sampled uniformly without replacement. Thus,

$$X \in \{0, 1, 2, 3\}.$$

For $k = 0, 1, 2, 3$,

$$P(X = k) = \frac{\binom{4}{k}\binom{6}{3-k}}{\binom{10}{3}}.$$

Since

$$\binom{10}{3} = 120,$$

we get

$$\begin{aligned} P(X=0) &= \frac{\binom{4}{0}\binom{6}{3}}{120} = \frac{20}{120} = \frac{1}{6}, \\ P(X=1) &= \frac{\binom{4}{1}\binom{6}{2}}{120} = \frac{60}{120} = \frac{1}{2}, \\ P(X=2) &= \frac{\binom{4}{2}\binom{6}{1}}{120} = \frac{36}{120} = \frac{3}{10}, \\ P(X=3) &= \frac{\binom{4}{3}\binom{6}{0}}{120} = \frac{4}{120} = \frac{1}{30}. \end{aligned}$$

Therefore,

$$p_X(k) = \begin{cases} \frac{1}{6}, & k=0, \\ \frac{1}{2}, & k=1, \\ \frac{3}{10}, & k=2, \\ \frac{1}{30}, & k=3, \\ 0, & \text{otherwise.} \end{cases}$$

Using this PMF,

$$\begin{aligned} \mathbb{E}[X] &= 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{30} \\ &= \frac{1}{2} + \frac{3}{5} + \frac{1}{10} \\ &= \frac{6}{5}. \end{aligned}$$

(b) For at least one red ball, it is easiest to use the complement:

$$P(\text{at least one red}) = 1 - P(\text{no red}).$$

The event of no red means all three selected balls are blue, so

$$P(\text{no red}) = \frac{\binom{6}{3}}{\binom{10}{3}} = \frac{20}{120} = \frac{1}{6}.$$

Thus,

$$P(\text{at least one red}) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Next, all three selected balls have the same color if all three are red or all three are blue. Therefore,

$$P(\text{all three have the same color}) = \frac{\binom{4}{3} + \binom{6}{3}}{\binom{10}{3}} = \frac{4+20}{120} = \frac{24}{120} = \frac{1}{5}.$$

(c) With replacement, each draw is red with probability

$$p = \frac{4}{10} = \frac{2}{5},$$

independently across the three draws. Thus,

$$Z \sim \text{Binomial}\left(3, \frac{2}{5}\right).$$

Therefore,

$$P(Z=k)=\binom{3}{k}\left(\frac{2}{5}\right)^k\left(\frac{3}{5}\right)^{3-k}, \qquad k=0,1,2,3.$$

Equivalently,

$$p_Z(k)=\begin{cases}\frac{27}{125}, & k=0,\\ \frac{54}{125}, & k=1,\\ \frac{36}{125}, & k=2,\\ \frac{8}{125}, & k=3,\\ 0, & \text{otherwise.}\end{cases}$$

Since $Z \sim \text{Binomial}(3, 2/5)$,

$$\text{Var}(Z)=3\cdot\frac{2}{5}\cdot\frac{3}{5}=\frac{18}{25}.$$

(d*) Let

$$p=P(\text{red})=\frac{2}{5}, \qquad q=P(\text{blue})=\frac{3}{5}.$$

It is impossible to terminate after only one draw, so

$$P(W=1)=0.$$

We terminate at $W=2$ if the first two draws have the same color:

$$P(W=2)=p^2+q^2=\left(\frac{2}{5}\right)^2+\left(\frac{3}{5}\right)^2=\frac{13}{25}.$$

We terminate at $W=3$ if the first two draws are different and the second and third draws have the same color:

$$P(W=3)=pq^2+qp^2=pq(p+q)=pq=\frac{6}{25}.$$

For the general PMF, termination at time $w \geq 2$ means that the first $w-1$ colors alternate, and the w-th color equals the $(w-1)$-st color. There are two possible alternating patterns, depending on whether the first draw is red or blue.

If $w=2m$ is even, then

$$P(W=2m)=p^{m+1}q^{m-1}+q^{m+1}p^{m-1}=p^{m-1}q^{m-1}(p^2+q^2), \qquad m=1,2,\ldots.$$

If $w=2m+1$ is odd, then

$$P(W=2m+1)=p^mq^{m+1}+q^mp^{m+1}=p^mq^m, \qquad m=1,2,\ldots.$$

Here $p=2/5$ and $q=3/5$.

Problem 4. Joint and conditional PMFs

(a) The marginal PMF of X is obtained by summing over y:

$$p_X(0)=\frac{1}{8}+\frac{1}{8}+\frac{1}{4}=\frac{1}{2}, \qquad p_X(1)=\frac{1}{4}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}.$$

Thus,

$$p_X(x)=\begin{cases}1/2, & x=0,\\ 1/2, & x=1,\\ 0, & \text{otherwise.}\end{cases}$$

The marginal PMF of Y is obtained by summing over x:

$$p_Y(0) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}, \qquad p_Y(1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}, \qquad p_Y(2) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}.$$

$$p_Y(y) = \begin{cases} 3/8, & y = 0, \\ 1/4, & y = 1, \\ 3/8, & y = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Also, the event $\{X < Y\}$ consists of three outcomes $\{(0,1), (0,2), (1,2)\}$, and thus,

$$P(X < Y) = p_{X,Y}(0,1) + p_{X,Y}(0,2) + p_{X,Y}(1,2) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}.$$

(b) For $x = 0$, we divide the row $x = 0$ by $p_X(0) = 1/2$:

$$p_{Y|X}(0 \mid 0) = \frac{1/8}{1/2} = \frac{1}{4}, \qquad p_{Y|X}(1 \mid 0) = \frac{1/8}{1/2} = \frac{1}{4}, \qquad p_{Y|X}(2 \mid 0) = \frac{1/4}{1/2} = \frac{1}{2}.$$

For $x = 1$, we divide the row $x = 1$ by $p_X(1) = 1/2$:

$$p_{Y|X}(0 \mid 1) = \frac{1/4}{1/2} = \frac{1}{2}, \qquad p_{Y|X}(1 \mid 1) = \frac{1/8}{1/2} = \frac{1}{4}, \qquad p_{Y|X}(2 \mid 1) = \frac{1/8}{1/2} = \frac{1}{4}.$$

Thus,

$$p_{Y|X}(y \mid 0) = \begin{cases} 1/4, & y = 0, \\ 1/4, & y = 1, \\ 1/2, & y = 2, \\ 0, & \text{otherwise,} \end{cases} \qquad , \qquad p_{Y|X}(y \mid 1) = \begin{cases} 1/2, & y = 0, \\ 1/4, & y = 1, \\ 1/4, & y = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Finally,

$$\begin{aligned} \mathbb{E}[Y \mid X = 0] &= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} = \frac{5}{4}, \\ \mathbb{E}[Y \mid X = 1] &= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

(c) Using the law of total expectation and the results from part (b),

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[Y \mid X = 0]P(X = 0) + \mathbb{E}[Y \mid X = 1]P(X = 1) \\ &= \frac{5}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = 1. \end{aligned}$$

Next,

$$\mathbb{E}[XY] = \sum_{x,y} xy \, p_{X,Y}(x,y).$$

Only the two pairs $(x,y) \in \{(1,1), (1,2)\}$ contributes to this weighted sum, so

$$\mathbb{E}[XY] = 1 \cdot 1 \cdot \frac{1}{8} + 1 \cdot 2 \cdot \frac{1}{8} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

The random variables X and Y are not independent. For example,

$$p_{X,Y}(0,0) = \frac{1}{8} \neq \frac{3}{16} = \frac{1}{2} \cdot \frac{3}{8} = p_X(0)p_Y(0).$$

Since, there exists at least one pair (x,y) for which the joint PMF does not factor as $p_X p_Y$, X and Y are not independent.

(d*) Let

$$q_y = P(X = 1, Y = y), \qquad y = 0, 1, 2.$$

The fixed marginals require

$$0 \le q_0 \le \frac{3}{8}, \qquad 0 \le q_1 \le \frac{1}{4}, \qquad 0 \le q_2 \le \frac{3}{8},$$

and

$$q_0 + q_1 + q_2 = P(X = 1) = \frac{1}{2}.$$

The mean of $X + Y$ is fixed by the marginals:

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{1}{2} + \left(0 \cdot \frac{3}{8} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8}\right) = \frac{3}{2}.$$

Thus, maximizing $\text{Var}(X + Y)$ is equivalent to maximizing

$$\mathbb{E}[(X + Y)^2].$$

For a fixed y, assigning mass q_y to $X = 1$ instead of $X = 0$ changes the squared value from y^2 to $(y+1)^2$, a gain of

$$(y + 1)^2 - y^2 = 2y + 1.$$

This gain is largest for $y = 2$, then $y = 1$, then $y = 0$. Therefore, to maximize the second moment, assign as much $X = 1$ mass as possible to the largest y's:

$$q_2 = \frac{3}{8}, \qquad q_1 = \frac{1}{8}, \qquad q_0 = 0.$$

This gives the joint PMF

$p_{X,Y}(x, y)$	$y = 0$	$y = 1$	$y = 2$
$x = 0$	$\frac{3}{8}$	$\frac{1}{8}$	0
$x = 1$	0	$\frac{1}{8}$	$\frac{3}{8}$

which has the required marginals.

For this joint PMF,

$$\begin{aligned} \mathbb{E}[(X + Y)^2] &= 0^2 \cdot \frac{3}{8} + 1^2 \cdot \frac{1}{8} + 2^2 \cdot \frac{1}{8} + 3^2 \cdot \frac{3}{8} \\ &= \frac{1}{8} + \frac{4}{8} + \frac{27}{8} \\ &= 4. \end{aligned}$$

Therefore,

$$\text{Var}(X + Y) = \mathbb{E}[(X + Y)^2] - \big(\mathbb{E}[X + Y]\big)^2 = 4 - \left(\frac{3}{2}\right)^2 = \frac{7}{4}.$$