STA 131A Introduction to Probability Theory

Midterm exam 2

Instructor: Dogyoon Song

Name: ____________________ **Student ID:** ____________________

Instructions: This midterm exam is **closed-book**, except for the permitted note sheet described below. You may bring a pen or pencil, one letter-sized sheet of *hand-written* notes (both sides), and a *non-graphing* calculator. No other materials (e.g., textbooks) are allowed. You have 50 minutes to complete all problems. The **total score is 120 points**, with *up to 10 bonus points available.* Once you receive this exam, please confirm that you have all 8 pages.

- Make sure to clearly write your name and student ID above.
- Present answers succinctly, but include all relevant steps for full credit. Partial credit is possible only if your reasoning is clearly shown and traceable by the grader.
- If necessary, round all numerical answers to three decimal places; you may leave fractions such as 8/55 unevaluated.
- Bonus problems can be more challenging; consider attempting them after you finish the main problems.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Total	

Problem 1 (30 points in total).

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} c(x+1), & 0 \le x \le 1, \\ c(3-x), & 1 < x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (10 points) Find the value of c, and briefly explain why f_X is a valid PDF.

(b) (10 points) Compute

$$P\left(\frac{1}{2} < X \le \frac{3}{2}\right).$$

(c) (10 points) Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Problem 2 (25 points in total).

Suppose

$$X \sim N(50, 8^2).$$

(a) (9 points) Compute

$$P(44 < X \leq 62).$$

Express your answer in terms of the standard normal CDF Φ. Then provide a numerical approximation.

(b) (8 points) Find a cutoff value q such that

$$P(X > q) = 0.10.$$

(c) (8 points) Let

$$Y = 2X + 5.$$

Find the distribution of Y, including its mean and variance.

Problem 3 (35 points in total + 5 bonus points).

Suppose we are modeling the time for a task that is completed in two stages. Let

$$X = \text{total completion time}, \qquad Y = \text{completion time of the first stage},$$

measured in appropriate units to have $0 \le Y \le X \le 1$. Suppose (X, Y) is jointly continuous with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c(x+y), & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (10 points) Find the value of c. Then find the marginal PDF of X.

(b) (10 points) For $0 < x \le 1$, find the conditional PDF $f_{Y|X}(y \mid x)$, and compute

$$\mathbb{E}[Y \mid X = x].$$

(c) (7 points) Determine whether X and Y are independent. Justify your answer.

(d) (8 points) Compute

$$P\left(Y \leq \frac{X}{2}\right).$$

(e*) (5 bonus points) Define

$$U = Y, \qquad V = X - Y,$$

so that U is the first-stage time and V is the second-stage time.

Find the joint PDF $f_{U,V}(u, v)$. Then determine whether U and V are independent. Justify your answer.

(*Hint:* Condition on $U = u$, so that $Y = u$ and $V = X - u$.)

Problem 4 (30 points in total + 5 bonus points).

(a) **(10 points)** Let

$$U \sim \text{Uniform}(0, 1), \qquad T = -\log U.$$

Find the CDF and PDF of T.

(b) **(10 points)** An unknown signal Θ is modeled as

$$\Theta \sim \text{Uniform}(-1, 1).$$

Then a noisy measurement Y is observed. Conditional on $\Theta = \theta$, suppose Y has the density

$$f_{Y|\Theta}(y \mid \theta) = \frac{1}{2}e^{-|y-\theta|}, \qquad y \in \mathbb{R}.$$

Compute

$$\mathbb{E}[Y \mid \Theta = \theta] \qquad \text{and} \qquad \mathbb{E}[Y].$$

(c) **(10 points)** Continue with the model from part **(b)**. Suppose that the observed measurement is

$$Y = 1.$$

Use Bayes' rule to find the posterior density $f_{\Theta|Y}(\theta \mid 1)$ for $-1 \leq \theta \leq 1$. Then compute

$$P(\Theta \geq 0 \mid Y = 1).$$

(d*) **(5 bonus points)** Let A and B be independent Exponential(1) random variables, and define

$$D = A - B.$$

Show that

$$f_D(d) = \frac{1}{2}e^{-|d|}, \qquad d \in \mathbb{R}.$$

In other words, the difference of two independent Exponential(1) random variables has the Laplace density used in part **(b)**. (*Hint:* Use convolution or any valid density calculation.)

Standard Normal Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990