

# STA 131A Introduction to Probability Theory

## (Practice Midterm 2 – Version A)

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**Instructions.** This practice midterm is designed to resemble a 50-minute in-class exam. However, the actual Midterm 2 may differ in content or style from this practice exam. Assume you may use one *handwritten*, two-sided, letter-sized cheat sheet and a simple non-graphing calculator. A standard normal table is included on the last page. The **total score is 120 points**.

- Make sure to clearly write your name and ID above.
- Present your answers clearly and show enough work to justify your conclusions for full credit.
- Partial credit is possible only if your reasoning is clearly shown and easy to trace.
- If you use a theorem, formula, or table value, make it clear.

Problem	Score
Problem 1	
Problem 2	
Problem 3	
Problem 4	
<b>Total</b>	

**Problem 1 (30 points total). PDF, CDF, and moments**

Let  $X$  be a continuous random variable with density

$$f_X(x) = \begin{cases} c, & 0 \leq x < 1, \\ 2c, & 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (10 points) Find the value of  $c$ . Then find the CDF  $F_X(x)$  for all real  $x$ .

(b) (10 points) Compute

$$P(0.5 < X \leq 1.5) \quad \text{and} \quad \mathbb{E}[X].$$

(c) (10 points) Compute  $\text{Var}(X)$ .

**Problem 2 (25 points total). Normal random variables**

Suppose

$$X \sim N(100, 15^2).$$

**(a) (9 points)** Compute

$$P(85 < X \leq 120).$$

Express your answer in terms of the standard normal CDF  $\Phi$ , and then give a numerical approximation using the normal table attached.

**(b) (8 points)** Find a cutoff  $c$  such that

$$P(X > c) = 0.05.$$

You may use the normal table attached.

**(c) (8 points)** Let

$$Y = 2X - 50.$$

Find the distribution of  $Y$ , including its mean and variance.

**Problem 3 (35 points total). Joint PDF**

Let  $(X, Y)$  be jointly continuous with joint density

$$f_{X,Y}(x, y) = \begin{cases} c, & 0 \leq x \leq 1, \quad x \leq y \leq x + 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (8 points) Sketch the support of  $(X, Y)$ , and find the value of  $c$ .

(b) (10 points) Find the marginal densities  $f_X(x)$  and  $f_Y(y)$ .

(c) (12 points) For  $0 < y < 2$ , find the conditional density  $f_{X|Y}(x | y)$ , and compute

$$\mathbb{E}[X | Y = y].$$

(d) (5 points) Determine whether  $X$  and  $Y$  are independent. Justify your answer.

**Problem 4 (30 points total). Transformations and conditional expectation**

(a) (10 points) Let  $X \sim \text{Uniform}(0, 1)$ , and define

$$Y = e^X.$$

Find the CDF and PDF of  $Y$ .

(b) (10 points) Let  $U \sim \text{Uniform}(0, 1)$ . Conditional on  $U = u$ , suppose

$$N | U = u \sim \text{Binomial}(10, u).$$

Compute

$$\mathbb{E}[N | U = u], \quad \mathbb{E}[N | U], \quad \mathbb{E}[N].$$

(c) (10 points) A hidden state  $H$  takes values  $A, B, C$ , with prior probabilities

$$P(H = A) = 0.2, \quad P(H = B) = 0.5, \quad P(H = C) = 0.3.$$

A continuous measurement  $Y$  is observed. At the observed value  $Y = y_0$ , the likelihood densities are

$$f_{Y|H}(y_0 | A) = 0.1, \quad f_{Y|H}(y_0 | B) = 0.3, \quad f_{Y|H}(y_0 | C) = 0.6.$$

Compute the posterior probabilities

$$P(H = A | Y = y_0), \quad P(H = B | Y = y_0), \quad P(H = C | Y = y_0).$$

## Standard Normal Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990