STA 131A Introduction to Probability Theory

(Practice Midterm 2 – Version A Solution)

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Problem 1

(a) Normalization gives

$$1 = \int_0^1 c\,dx + \int_1^2 2c\,dx = c + 2c = 3c,$$

so $c = 1/3$. Therefore

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x/3, & 0 \le x < 1, \\ \frac{1}{3} + \int_1^x \frac{2}{3}\,dt = \frac{2x-1}{3}, & 1 \le x \le 2, \\ 1, & x > 2. \end{cases}$$

(b)

$$P(0.5 < X \le 1.5) = F_X(1.5) - F_X(0.5) = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}.$$

Also,

$$\mathbb{E}[X] = \int_0^1 x\frac{1}{3}\,dx + \int_1^2 x\frac{2}{3}\,dx = \frac{1}{6} + 1 = \frac{7}{6}.$$

(c)

$$\mathbb{E}[X^2] = \int_0^1 x^2\frac{1}{3}\,dx + \int_1^2 x^2\frac{2}{3}\,dx = \frac{1}{9} + \frac{14}{9} = \frac{5}{3}.$$

Hence

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{5}{3} - \left(\frac{7}{6}\right)^2 = \frac{11}{36}.$$

Problem 2

(a) Standardizing,

$$P(85 < X \le 120) = P\left(\frac{85-100}{15} < Z \le \frac{120-100}{15}\right) = P(-1 < Z \le 1.33\overline{3}).$$

Thus

$$P(85 < X \le 120) = \Phi(1.33\overline{3}) - \Phi(-1).$$

Using the table, $\Phi(1.33) \approx 0.9082$ and $\Phi(-1) = 0.1587$, so

$$P(85 < X \le 120) \approx 0.9082 - 0.1587 = 0.7495.$$

(b) We need $P(X \le c) = 0.95$. Since $\Phi(1.645) \approx 0.95$,

$$\frac{c-100}{15} = 1.645.$$

Therefore

$$c = 100 + 15(1.645) = 124.675.$$

(c) Since $Y = 2X - 50$, normality is preserved under linear transformations:

$$Y \sim N(2 \cdot 100 - 50,\ 2^2 \cdot 15^2) = N(150, 900).$$

Thus $\mathbb{E}[Y] = 150$ and $\mathrm{Var}(Y) = 900$.

Problem 3

(a) The support is the parallelogram

$$0 \le x \le 1, \qquad x \le y \le x+1.$$

For each fixed $x \in [0,1]$, the vertical length is 1, so the area is 1. Hence

$$c = 1.$$

(b) For $0 \le x \le 1$,

$$f_X(x) = \int_x^{x+1} 1\,dy = 1.$$

Thus

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

To find f_Y, rewrite the support as

$$\max(0, y-1) \le x \le \min(1, y).$$

Hence

$$f_Y(y) = \begin{cases} \int_0^y 1\,dx = y, & 0 \le y \le 1, \\ \int_{y-1}^1 1\,dx = 2-y, & 1 < y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(c) For $0 < y < 1$, the allowed values are $0 \le x \le y$, so

$$f_{X|Y}(x \mid y) = \frac{1}{y}, \qquad 0 \le x \le y.$$

Thus $X \mid Y = y \sim \text{Uniform}(0, y)$, and

$$\mathbb{E}[X \mid Y = y] = \frac{y}{2}.$$

For $1 < y < 2$, the allowed values are $y - 1 \le x \le 1$, so

$$f_{X|Y}(x \mid y) = \frac{1}{2-y}, \qquad y-1 \le x \le 1.$$

Thus $X \mid Y = y \sim \text{Uniform}(y-1, 1)$, and

$$\mathbb{E}[X \mid Y = y] = \frac{(y-1)+1}{2} = \frac{y}{2}.$$

(d) The random variables are not independent. For example, the support is not rectangular. Also, on the support, $f_{X,Y}(x,y) = 1$, while

$$f_X(x)f_Y(y) = f_Y(y),$$

which is not identically equal to 1.

Problem 4

(a) Since $Y = e^X$, the possible values are $1 < Y < e$. For $1 \le y \le e$,

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \log y) = \log y.$$

Therefore

$$F_Y(y) = \begin{cases} 0, & y < 1, \\ \log y, & 1 \le y \le e, \\ 1, & y > e. \end{cases}$$

Differentiating,

$$f_Y(y) = \begin{cases} 1/y, & 1 < y < e, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Given $U = u$, $N \sim \text{Binomial}(10, u)$. Hence

$$\mathbb{E}[N \mid U = u] = 10u.$$

Therefore

$$\mathbb{E}[N \mid U] = 10U.$$

By the law of iterated expectation,

$$\mathbb{E}[N] = \mathbb{E}[10U] = 10\mathbb{E}[U] = 5.$$

(c) The normalizing denominator is

$$0.2 \times 0.1 + 0.5 \times 0.3 + 0.3 \times 0.6 = 0.02 + 0.15 + 0.18 = 0.35.$$

Thus

$$P(H = A \mid Y = y_0) = \frac{0.02}{0.35} = \frac{2}{35},$$
$$P(H = B \mid Y = y_0) = \frac{0.15}{0.35} = \frac{3}{7},$$
$$P(H = C \mid Y = y_0) = \frac{0.18}{0.35} = \frac{18}{35}.$$