STA 131A Introduction to Probability Theory

(Practice Midterm 2 – Version B)

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Instructions. This practice midterm is designed to resemble a 50-minute in-class exam. However, the actual Midterm 2 may differ in content or style from this practice exam. Assume you may use one *hand-written*, two-sided, letter-sized cheat sheet and a simple non-graphing calculator. A standard normal table is included on the last page. The **total score is 120 points**.

- Make sure to clearly write your name and ID above.
- Present your answers clearly and show enough work to justify your conclusions for full credit.
- Partial credit is possible only if your reasoning is clearly shown and easy to trace.
- If you use a theorem, formula, or table value, make it clear.

Problem	**Score**
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Total	

Problem 1 (30 points total). CDF, PDF, and moments

Let X be a continuous random variable with CDF

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x^2, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

(a) (10 points) Find the PDF $f_X(x)$.

(b) (8 points) Compute

$$P\left(\frac{1}{4} < X \le \frac{3}{4}\right).$$

(c) (12 points) Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Problem 2 (25 points total). Normal random variables and expected counts

Suppose product lifetimes, measured in hours, satisfy

$$X \sim N(40, 6^2).$$

(a) (8 points) Find the 10-th percentile q, i.e., find q such that

$$P(X \leq q) = 0.10.$$

You may use the normal table attached.

(b) (9 points) A product is called acceptable if its lifetime exceeds the cutoff q from part **(a)**. Among 50 independent products, let

$$N = \text{the number of acceptable products.}$$

Identify the distribution of N, and compute $\mathbb{E}[N]$ and $\text{Var}(N)$.

(c) (8 points) Suppose lifetime is instead recorded in minutes:

$$Y = 60X.$$

Find the distribution of Y, including its mean and standard deviation.

Problem 3 (35 points total). Joint PDF with nonconstant density

Let (X, Y) be jointly continuous with joint density

$$f_{X,Y}(x, y) = \begin{cases} cy, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (12 points) Find the value of c. Then find the marginal densities $f_X(x)$ and $f_Y(y)$.

(b) (10 points) Find the conditional density $f_{Y|X}(y \mid x)$, and compute

$$\mathbb{E}[Y \mid X = x].$$

(c) (8 points) Compute

$$P(Y \le X/2).$$

(d) (5 points) Determine whether X and Y are independent. Justify your answer.

Problem 4 (30 points total). Derived distributions and conditional expectation

(a) (10 points) Let $X \sim \text{Uniform}(0, 1)$, and let

$$Y = \sqrt{X}.$$

Find the CDF and PDF of Y.

(b) (10 points) Let $\Lambda \sim \text{Uniform}(1, 2)$. Conditional on $\Lambda = \lambda$, suppose

$$T \mid \Lambda = \lambda \sim \text{Exponential}(\lambda).$$

Compute

$$\mathbb{E}[T \mid \Lambda = \lambda], \qquad \mathbb{E}[T \mid \Lambda], \qquad \mathbb{E}[T].$$

(c) (10 points) In the setting of part **(b)**, write the posterior density

$$f_{\Lambda|T}(\lambda \mid t)$$

for $1 \leq \lambda \leq 2$, up to the appropriate normalizing constant. Then write the fully normalized expression.

Standard Normal Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990