STA 131A Introduction to Probability Theory

(Practice Midterm 2 – Version B Solution)

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Problem 1

(a) Differentiating the CDF on $(0,1)$,

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$P\left(\frac{1}{4} < X \le \frac{3}{4}\right) = F_X\left(\frac{3}{4}\right) - F_X\left(\frac{1}{4}\right) = \frac{9}{16} - \frac{1}{16} = \frac{1}{2}.$$

(c)

$$\mathbb{E}[X] = \int_0^1 x(2x)\,dx = \frac{2}{3}.$$

$$\mathbb{E}[X^2] = \int_0^1 x^2(2x)\,dx = \frac{1}{2}.$$

Hence

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

Problem 2

(a) We need

$$P(X \le q) = 0.10.$$

Since $\Phi(-1.282) \approx 0.10$,

$$\frac{q - 40}{6} = -1.282.$$

Therefore

$$q = 40 + 6(-1.282) = 32.308.$$

(b) Since $P(X > q) = 0.90$,

$$N \sim \text{Binomial}(50, 0.90).$$

Hence

$$\mathbb{E}[N] = 50(0.90) = 45, \qquad \mathrm{Var}(N) = 50(0.90)(0.10) = 4.5.$$

(c) Since $Y = 60X$,

$$Y \sim N(60 \cdot 40,\ 60^2 \cdot 6^2) = N(2400, 129600).$$

Thus $\mathbb{E}[Y] = 2400$ and the standard deviation of Y is

$$60 \cdot 6 = 360.$$

Problem 3

(a) Normalization gives

$$1 = \int_0^1 \int_0^x cy\,dy\,dx = \int_0^1 \frac{cx^2}{2}\,dx = \frac{c}{6}.$$

Hence $c = 6$. Thus

$$f_{X,Y}(x,y) = 6y, \qquad 0 \le y \le x \le 1.$$

The marginal density of X is

$$f_X(x) = \int_0^x 6y\,dy = 3x^2, \qquad 0 \le x \le 1.$$

The marginal density of Y is

$$f_Y(y) = \int_y^1 6y\,dx = 6y(1-y), \qquad 0 \le y \le 1.$$

(b) For $0 < x \le 1$ and $0 \le y \le x$,

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{6y}{3x^2} = \frac{2y}{x^2}.$$

Therefore

$$\mathbb{E}[Y \mid X = x] = \int_0^x y\frac{2y}{x^2}\,dy = \frac{2}{x^2} \cdot \frac{x^3}{3} = \frac{2x}{3}.$$

(c)

$$P(Y \le X/2) = \int_0^1 \int_0^{x/2} 6y\,dy\,dx = \int_0^1 3\left(\frac{x}{2}\right)^2 dx = \int_0^1 \frac{3x^2}{4}\,dx = \frac{1}{4}.$$

(d) The random variables are not independent. For example, the support is triangular, not rectangular. Equivalently,

$$f_{X,Y}(x,y) = 6y$$

does not factor as $f_X(x)f_Y(y) = 3x^2 \cdot 6y(1-y)$ on the support.

Problem 4

(a) Since $Y = \sqrt{X}$, the possible values are $0 < Y < 1$. For $0 \le y \le 1$,

$$F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = y^2.$$

Hence

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ y^2, & 0 \le y \le 1, \\ 1, & y > 1. \end{cases}$$

Differentiating,

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Given $\Lambda = \lambda$, $T \sim \text{Exponential}(\lambda)$, so

$$\mathbb{E}[T \mid \Lambda = \lambda] = \frac{1}{\lambda}.$$

Therefore

$$\mathbb{E}[T \mid \Lambda] = \frac{1}{\Lambda}.$$

By the law of iterated expectation,

$$\mathbb{E}[T] = \mathbb{E}\left[\frac{1}{\Lambda}\right] = \int_1^2 \frac{1}{\lambda}\, d\lambda = \log 2.$$

(c) Since $f_\Lambda(\lambda) = 1$ for $1 \le \lambda \le 2$, and

$$f_{T|\Lambda}(t \mid \lambda) = \lambda e^{-\lambda t}, \qquad t \ge 0,$$

Bayes' rule gives

$$f_{\Lambda|T}(\lambda \mid t) \propto \lambda e^{-\lambda t}, \qquad 1 \le \lambda \le 2.$$

The normalized posterior density is

$$f_{\Lambda|T}(\lambda \mid t) = \frac{\lambda e^{-\lambda t}}{\int_1^2 s e^{-st}\, ds}, \qquad 1 \le \lambda \le 2.$$