STA 131A – Homework 3

Submission due: Tue, April 21 at 11:59 PM PT

Instructor: Dogyoon Song

Instructions: Upload a single PDF file to Gradescope via Canvas ("Homework 3" under "Assignments"). Name the file using the prefix of your UC Davis email ID and the homework number (e.g., `dgsong_hw3.pdf`). Include "STA 131A," your name, and the last four digits of your student ID on the front page. No late submissions will be accepted; any submission received after the deadline will receive 0 points. For full information about submission requirements and the late submission policy, see the syllabus.

Problem 1 (30 points in total).

(a) (5 points) A student starts with 3 bonus points. The student then answers 3 true/false questions, each independently correct with probability 2/3. Each correct answer earns 2 additional points, whereas each incorrect answer deducts 1 point. Let X denote the student's total number of points after all 3 questions have been answered. Find the PMF of X.

(b) (10 points) The joint PMF of two discrete random variables X and Y is given by

$p_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	1/8	1/8	1/4
$x=1$	1/4	1/8	1/8

and $p_{X,Y}(x,y)=0$ for all other pairs (x,y).

(i) Find the marginal PMFs p_X and p_Y.

(ii) Compute $P(X<Y)$.

(iii) Compute $P(Y=2 \mid X=0)$.

(c) (7 points) Let X be a discrete random variable with $\mathbb{E}[X^2]<\infty$, and let $a\in\mathbb{R}$. Prove that

$$\mathbb{E}\big[(X-a)^2\big] = \mathrm{Var}(X) + \big(\mathbb{E}[X]-a\big)^2.$$

Thereafter, determine the value of $a\in\mathbb{R}$ that minimizes $\mathbb{E}\big[(X-a)^2\big]$.

(d) (8 points) Let X and Y be discrete random variables with joint PMF $p_{X,Y}$.

(i) Prove the law of total probability in PMF form:

$$p_X(x) = \sum_y p_{X|Y}(x \mid y)\, p_Y(y), \qquad x\in\mathbb{R},$$

where the sum is over all values y that Y can take.

(ii) Deduce Bayes' rule in PMF form, which states that for any x, y with $p_X(x)>0$ and $p_Y(y)>0$,

$$p_{Y|X}(y \mid x) = \frac{p_{X|Y}(x \mid y)\, p_Y(y)}{p_X(x)}.$$

Problem 2 (25 points in total).

(a) (7 points) [BT08, Chapter 2, Problem 17, p. 122]

(b) (8 points) [BT08, Chapter 2, Problem 21, p. 123]

(c) (10 points) [BT08, Chapter 2, Problem 23, pp. 123–124]

Problem 3 (20 points in total).

(a) (10 points) [BT08, Chapter 2, Problem 24, p. 124]

(b) (10 points) [BT08, Chapter 2, Problem 26, p. 124]

Problem 4 (25 points in total).

(a) (5 points) [BT08, Chapter 2, Problem 31, p. 128]

(b) (10 points) [BT08, Chapter 2, Problem 32, p. 128]

(c) (10 points) [BT08, Chapter 2, Problem 33, p. 128]

Problem 5* (up to 10 bonus points).

Each subproblem is worth 5 bonus points, and at most two bonus subproblems will count, for a maximum of 10 bonus points.

(a) (5 bonus points) [BT08, Chapter 2, Problem 14, p. 122]

(b) (5 bonus points) [BT08, Chapter 2, Problem 22, p. 123]

(c) (5 bonus points) [BT08, Chapter 2, Problem 40, p. 133]

References

[BT08] Dimitri Bertsekas and John N Tsitsiklis. *Introduction to probability*, volume 1. Athena Scientific, 2nd edition, 2008.