STA 131A – Homework 4

Submission due: Tue, May 5 at 11:59 PM PT

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Instructions: Upload a single PDF file to Gradescope via Canvas ("Homework 4" under "Assignments"). Name the file using the prefix of your UC Davis email ID and the homework number (e.g., `dgsong_hw4.pdf`). Include "STA 131A," your name, and the last four digits of your student ID on the front page. No late submissions will be accepted; any submission received after the deadline will receive 0 points. For full information about submission requirements and the late submission policy, see the syllabus.

Problem 1 (25 points in total). Investment portfolio under market uncertainty

The market can be in one of three possible states during the coming year:

$$M_1 = \{\text{prosperity}\}, \qquad M_2 = \{\text{normal year}\}, \qquad M_3 = \{\text{recession}\}.$$

Suppose

$$P(M_1) = \frac{1}{4}, \qquad P(M_2) = \frac{7}{12}, \qquad P(M_3) = \frac{1}{6}.$$

Let R denote the annual portfolio return, measured in percentage points. Thus, $R = 10$ means a return of 10%. Let

$$B = \{R \geq 10\}$$

denote the event that the annual return is at least 10%.

Suppose also that

$$P(B \mid M_1) = 0.9, \qquad P(B \mid M_2) = 0.7, \qquad P(B \mid M_3) = 0.2.$$

(a) **(5 points)** Compute $P(B)$, the probability that the annual return is at least 10%.

(b) **(5 points)** Given the observation of B, compute the posterior probabilities

$$P(M_1 \mid B), \qquad P(M_2 \mid B), \qquad P(M_3 \mid B).$$

Which market state is most likely, given B?

(c) **(5 points)** For each market state, suppose that the portfolio return R has the following conditional distribution:

$$P(R = r \mid M_1) = \begin{cases} 0.2, & r = 30, \\ 0.3, & r = 20, \\ 0.4, & r = 15, \\ 0.1, & r = 5, \\ 0, & \text{otherwise}, \end{cases}$$

$$P(R = r \mid M_2) = \begin{cases} 0.2, & r = 20, \\ 0.5, & r = 10, \\ 0.3, & r = 5, \\ 0, & \text{otherwise}, \end{cases}$$

$$P(R = r \mid M_3) = \begin{cases} 0.2, & r = 10, \\ 0.5, & r = -10, \\ 0.3, & r = -40, \\ 0, & \text{otherwise}. \end{cases}$$

Compute the conditional expectation $\mathbb{E}[R \mid M_i]$ for each $i \in \{1, 2, 3\}$.

(d) **(5 points)** Use the law of total expectation to compute the unconditional expected return $\mathbb{E}[R]$.

(e) **(5 points)** Compute the conditional expectation

$$\mathbb{E}[R \mid B].$$

Hint: For each market state M_i, first compute $\mathbb{E}[R \mid M_i, B]$, and then use the posterior probabilities from part **(b)**.

Problem 2 (35 points in total + 5 bonus points). Sampling from an urn

An urn contains 3 white balls and 7 black balls. Unless otherwise stated, treat the 10 balls as individually distinguishable.

(a) (9 points) Suppose first that three balls are drawn sequentially without replacement.

- (i) How many ordered sequences of 3 distinct balls are possible, assuming that the 10 balls are individually distinguishable?
- (ii) How many ordered length-3 color sequences are possible?
- (iii) How many ordered sequences of distinct balls from part (i) produce the color pattern (Black, Black, Black)? How many produce the color pattern (White, Black, White)?

(b) (9 points) Now suppose that five balls are drawn sequentially without replacement.

- (i) How many unordered 5-subsets of distinct balls are possible?
- (ii) How many ways are there to split the 10 balls into two *unlabeled* groups of 5 balls each?
- (iii) How many different color compositions are possible for a 5-ball selection?
  Hint: How many different values can the number of white balls among the 5 selected balls take?

(c) (8 points) Now suppose that three balls are selected uniformly at random without replacement, meaning that every 3-ball subset of the 10 individual balls is equally likely to be chosen. Let

$$X = \text{the number of white balls selected.}$$

Find the PMF of X, and compute $\mathbb{E}[X]$.

Hint: For $k = 0, 1, 2, 3$, count the number of ways to choose k white balls and $3 - k$ black balls.

(d) (9 points) Now suppose the labels $0, 1, \ldots, 9$ are assigned to the balls, and that balls of the same color are regarded as *indistinguishable* once the labels are assigned. Equivalently, a labeling is completely determined by choosing which 3 of the 10 labels are assigned to the white balls.

- (i) How many distinct labelings are possible?
- (ii) If all distinct labelings are equally likely, what is the probability that the white balls receive the labels $\{3, 6, 9\}$?
- (iii) What is the probability that all three white balls receive odd labels?

(e*) (5 bonus points) Now consider a new urn containing 3 red balls, 5 yellow balls, and 7 green balls. Unless otherwise stated, treat the 15 balls as individually distinguishable.

- (i) Suppose 5 balls are selected without replacement, and only the *color composition* matters. How many different color compositions are possible?
  Hint: Equivalently, how many triples (r, y, g) of nonnegative integers satisfy

$$r + y + g = 5, \qquad 0 \le r \le 3, \quad 0 \le y \le 5, \quad 0 \le g \le 7?$$

- (ii) Suppose that all 15 balls are split uniformly at random into three *unlabeled* groups of 5 balls each.
  - (a) How many such groupings are possible?
  - (b) In how many of these groupings do the three red balls lie in three different groups?

Problem 3 (20 points in total + 5 bonus points). Joint and conditional PMFs

The joint PMF of two discrete random variables X and Y is

$p_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	c	c	0
$x=1$	c	$2c$	c
$x=2$	0	c	c

and $p_{X,Y}(x,y)=0$ for all other pairs.

(a) **(6 points)** Find the value of c. Then compute the marginal PMFs p_X and p_Y.

(b) **(6 points)** For each $x \in \{0,1,2\}$, compute the conditional PMF $p_{Y|X}(y \mid x)$ and the conditional expectation $\mathbb{E}[Y \mid X=x]$.

(c) **(8 points)** Use the law of total expectation to compute $\mathbb{E}[Y]$. Then compute $\mathrm{Var}(Y)$. Finally, determine whether X and Y are independent, and justify your answer.

(d*) **(5 bonus points)** For each $y \in \{0,1,2\}$, compute

$$\mathbb{E}[X \mid Y=y] \qquad \text{and} \qquad \mathrm{Var}(X \mid Y=y).$$

Then use conditioning on Y to compute $\mathbb{E}[XY]$. Based on your answer, determine whether the identity

$$\mathbb{E}[XY]=\mathbb{E}[X]\mathbb{E}[Y]$$

holds in this example.

Problem 4 (25 points in total).

(a) **(5 points)** [BT08, Chapter 3, Problem 2, p. 184]

(b) **(5 points)** [BT08, Chapter 2, Problem 6, p. 186]

(c) **(10 points)** [BT08, Chapter 2, Problem 7, p. 186]

References

[BT08] Dimitri Bertsekas and John N Tsitsiklis. *Introduction to probability*, volume 1. Athena Scientific, 2nd edition, 2008.