STA 131A – Homework 5

Submission due: Tue, May 12 at 11:59 PM PT

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Instructions: Upload a single PDF file to Gradescope via Canvas ("Homework 5" under "Assignments"). Name the file using the prefix of your UC Davis email ID and the homework number (e.g., `dgsong_hw5.pdf`). Include "STA 131A," your name, and the last four digits of your student ID on the front page. No late submissions will be accepted; any submission received after the deadline will receive 0 points. For full information about submission requirements and the late submission policy, see the syllabus.

Problem 1 (20 points in total). Normal random variables and standardization

(a) (8 points) [BT08, Chapter 3, Problem 11, p. 188]

(b) (6 points) [BT08, Chapter 3, Problem 12, p. 188]

(c) (6 points) [BT08, Chapter 3, Problem 13, p. 189]

Problem 2 (30 points in total). Jointly continuous random variables

(a) (10 points) Suppose that X and Y are jointly continuous with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c(x+2y), & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

for some constant $c > 0$.

(i) Find the value of c.

(ii) Find the marginal PDFs f_X and f_Y, including their supports.

(iii) Compute $P(X + Y \le 1)$.

(b) (10 points) Suppose that (X, Y) is uniformly distributed over the triangular region

$$T = \{(x,y) : 0 \le x \le 1 \text{ and } 0 \le y \le 3x\}.$$

That is, there exists some constant $c > 0$ such that

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in T, \\ 0, & \text{otherwise}. \end{cases}$$

(i) Find the value of c, and then find the marginal PDFs f_X and f_Y.

(ii) Compute $P(X + Y \ge 1)$.

(iii) Compute $\mathbb{E}[X + 2Y]$.

(c) (10 points) [BT08, Chapter 3, Problem 15, pp. 189–190]

Problem 3 (35 points in total). Conditional PDFs and conditional expectation

(a) (10 points) [BT08, Chapter 3, Problem 18, p. 190]

(b) (15 points) [BT08, Chapter 3, Problem 23, p. 191]

(c) (10 points) [BT08, Chapter 3, Problem 34, p. 198]

Problem 4 (15 points in total). Derived distributions

(a) (5 points) [BT08, Chapter 4, Problem 1, p. 246]

(b) (5 points) [BT08, Chapter 4, Problem 2, p. 246]

(c) (5 points) [BT08, Chapter 4, Problem 4, p. 246]

Problem 5* (up to 10 bonus points).

Each subproblem is worth 5 bonus points, and at most two bonus subproblems will count, for a maximum of 10 bonus points. In your submission, clearly identify at most two bonus subproblems you want graded.

(a) (5 bonus points) Let $Z \sim N(0,1)$, and define

$$W = Z^2.$$

Find the CDF and PDF of W.

Hint: For $w \geq 0$,

$$\{W \leq w\} = \{-\sqrt{w} \leq Z \leq \sqrt{w}\}.$$

(b) (5 bonus points) Let U and V be independent Uniform$(0,1)$ random variables, and define

$$D = |U - V|.$$

Find the CDF and PDF of D, and compute $\mathbb{E}[D]$.

Hint: It may be easier to compute $P(D \leq d)$ geometrically in the unit square.

(c) (5 bonus points) Let $X \sim$ Exponential(λ) and $Y \sim$ Exponential(μ), independently, where $\lambda, \mu > 0$. Define

$$T = \min\{X, Y\}.$$

Find the CDF and PDF of T. Then compute

$$P(X < Y).$$

Hint: Use

$$P(T > t) = P(X > t,\ Y > t).$$

References

[BT08] Dimitri Bertsekas and John N Tsitsiklis. *Introduction to probability*, volume 1. Athena Scientific, 2nd edition, 2008.