STA 131A – Homework 6

Submission due: Tue, May 26 at 11:59 PM PT

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Instructions: Upload a single PDF file to Gradescope via Canvas ("Homework 6" under "Assignments"). Name the file using the prefix of your UC Davis email ID and the homework number (e.g., `dgsong_hw6.pdf`). Include "STA 131A," your name, and the last four digits of your student ID on the front page. No late submissions will be accepted; any submission received after the deadline will receive 0 points. For full information about submission requirements and the late submission policy, see the syllabus.

Problem 1 (60 points in total). Midterm 2 review

In this problem, you will revisit the main problems from Midterm 2. Solve each problem carefully, and for each major part **(a)–(d)**, include a brief reflection of 1–3 sentences.

Your reflection should address the following:

- what you missed, found confusing, or could have done more efficiently during the exam;
- what concept or technique you reviewed afterward;
- how you would approach the problem now.

If you solved a part correctly on the exam, briefly explain how you checked your work or how you could solve it more efficiently.

Credit for this problem requires both mathematical work and a meaningful reflection. Reflections do not need to be long, but they should be specific.

(a) Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} c(x+1), & 0 \le x \le 1, \\ c(3-x), & 1 < x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(i) (5 points) Find the value of c, and briefly explain why f_X is a valid PDF.

(ii) (5 points) Compute

$$P\left(\frac{1}{2} < X \le \frac{3}{2}\right).$$

(iii) (5 points) Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

(b) Suppose

$$X \sim N(50, 8^2).$$

(i) **(4 points)** Compute

$$P(44 < X \le 62).$$

Express your answer using the standard normal CDF Φ. Then provide a numerical approximation.

(ii) **(4 points)** Find a cutoff value q such that

$$P(X > q) = 0.10.$$

(iii) **(4 points)** Let

$$Y = 2X + 5.$$

Find the distribution of Y, including its mean and variance.

(c) Suppose we are modeling the time for a task that is completed in two stages. Let

$$X = \text{total completion time}, \qquad Y = \text{completion time of the first stage},$$

measured in appropriate units to have $0 \le Y \le X \le 1$. Suppose (X, Y) is jointly continuous with joint PDF

$$f_{X,Y}(x, y) = \begin{cases} c(x + y), & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) **(5 points)** Find the value of c. Then find the marginal PDF of X.

(ii) **(5 points)** For $0 < x \le 1$, find the conditional PDF $f_{Y|X}(y \mid x)$, and compute

$$\mathbb{E}[Y \mid X = x].$$

(iii) **(4 points)** Determine whether X and Y are independent. Justify your answer.

(iv) **(4 points)** Compute

$$P\left(Y \le \frac{X}{2}\right).$$

(d) **(i)** **(5 points)** Let

$$U \sim \text{Uniform}(0, 1), \qquad T = -\log U.$$

Find the CDF and PDF of T.

(ii) **(5 points)** An unknown signal Θ is modeled as

$$\Theta \sim \text{Uniform}(-1, 1).$$

Then a noisy measurement Y is observed. Conditional on $\Theta = \theta$, suppose Y has the density

$$f_{Y|\Theta}(y \mid \theta) = \frac{1}{2} e^{-|y-\theta|}, \qquad y \in \mathbb{R}.$$

Compute

$$\mathbb{E}[Y \mid \Theta = \theta] \qquad \text{and} \qquad \mathbb{E}[Y].$$

(iii) **(5 points)** Continue with the model from part **(b)**. Suppose that the observed measurement is

$$Y = 1.$$

Use Bayes' rule to find the posterior density $f_{\Theta|Y}(\theta \mid 1)$ for $-1 \le \theta \le 1$. Then compute

$$P(\Theta \ge 0 \mid Y = 1).$$

Problem 2 (20 points in total + 5 bonus points).

(a)(i) **(3 points)** Suppose

$$\mathrm{Var}(X) = 4, \qquad \mathrm{Var}(Y) = 9, \qquad \mathrm{Cov}(X, Y) = -3.$$

Compute $\rho(X, Y)$, $\mathrm{Var}(X + Y)$, and $\mathrm{Var}(2X - Y)$.

(ii) **(3 points)** Let X be uniform on $\{-1, 0, 1\}$, and let $Y = X^2$. Compute $\mathrm{Cov}(X, Y)$. Are X and Y independent? Briefly justify your answer.

(iii) **(4 points)** Let $G \sim \mathrm{Bernoulli}(1/2)$, and suppose

$$X \mid G = 0 \sim N(0, 1), \qquad X \mid G = 1 \sim N(2, 4).$$

Compute

$$\mathbb{E}[\mathrm{Var}(X \mid G)], \qquad \mathrm{Var}(\mathbb{E}[X \mid G]),$$

and use the law of total variance to compute $\mathrm{Var}(X)$.

(b) (5 points) [BT08, Chapter 4, Problem 17]

(c) (5 points) [BT08, Chapter 4, Problem 18]

(d) (5 bonus points) [BT08, Chapter 4, Problem 19]

Problem 3 (20 points in total + 5 bonus points).

(a)(i) **(3 points)** Let X have PMF

$$P(X = 0) = \frac{1}{4}, \qquad P(X = 1) = \frac{1}{2}, \qquad P(X = 2) = \frac{1}{4}.$$

Find $M_X(t)$, and use it to compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

(ii) **(3 points)** Suppose a random variable Y has MGF

$$M_Y(t) = \exp(2t + 3t^2).$$

Compute $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

(iii) **(4 points)** Let $Z \sim \mathrm{Poisson}(\lambda)$. Use the MGF of Z to find the MGF of

$$W = 2Z + 3.$$

Then compute $\mathbb{E}[W]$.

(b) (5 points) Derive the MGF of the standard normal random variable.

(c) (5 points) [BT08, Chapter 4, Problem 30]

(d) (5 bonus points) [BT08, Chapter 4, Problem 42]

References

[BT08] Dimitri Bertsekas and John N Tsitsiklis. *Introduction to probability*, volume 1. Athena Scientific, 2nd edition, 2008.