# STA 131A: Introduction to Probability Theory

## Lecture 4: Independence

Dogyoon Song

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# Announcements

**Homework 1 is due tomorrow (Tue, Apr 7, 11:59 PM)**

- Please submit on time and follow the submission instructions
- TA office hours: Tue, 3:00 PM – 5:00 PM (MSB 1117)
- Feel free to post questions on Piazza (the earlier, the better)

# Agenda

**Last time:**

- Conditional probability
- Law of total probability
- Bayes' rule

**Today:** Independence

- Independence of two events
- Conditional independence
- Independence of several events

# Recap: Conditional probability and Bayes' rule

For an event[1] $B$ with $P(B) > 0$, the **conditional probability** of $A$ given $B$ is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

**Law of total probability:** If $\{A_1, \ldots, A_n\}$ is a partition[2] of $\Omega$, then for any event $B$,

$$P(B) = \sum_{i=1}^{n} P(A_i \cap B) = \sum_{i=1}^{n} P(A_i) \cdot P(B \mid A_i)$$

**Bayes' rule** states that *Posterior* = *Likelihood* × *Prior* / *Evidence*, i.e.,

$$P(A_i \mid B) = \frac{P(B \mid A_i)\, P(A_i)}{P(B)} \qquad \text{where} \qquad P(B) = \sum_{i=1}^{n} P(B \mid A_i) P(A_i)$$

[1] In this course, we only condition on events with positive probability

[2] such that $P(A_i) > 0$ for all $i$

# Recap example: Fair or double-headed?

## Example (Fair or double-headed?)

Let $A = \{\text{the coin is double-headed}\}$, so $A^c = \{\text{the coin is fair}\}$

Let $B = \{\text{the first toss is Head}\}$

- Suppose $P(A) = 0.5$
- $P(B \mid A) = 1$ (the double-headed coin always lands Head)
- $P(B \mid A^c) = 0.5$ (the fair coin lands Head with probability $1/2$)

**Q:** After observing one Head, how likely is it that the coin is double-headed?

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)}$$
$$= \frac{1 \times 0.5}{1 \times 0.5 + 0.5 \times 0.5} = \frac{2}{3} \approx 0.667.$$

- One Head raises the probability from 50% to 66.7%
- Data increase the probability of the model that better explains the observation

**Follow-up:** How would the posterior change after observing another Head or a Tail after a Head?

# Independence: Definition

Sometimes knowing that $B$ occurred does not change the probability of $A$

- When $P(B) > 0$, this means $P(A \mid B) = P(A)$

**Definition (Independence)**

The events $A$ and $B$ are **independent** if

$$P(A \cap B) = P(A)\,P(B)$$

- Independence is symmetric: $A$ is independent of $B \implies B$ is independent of $A$
  - We can unambiguously say "A and B are independent events"
- Events from distinct, non-interacting experiments are often modeled as independent
- Independence is very different from disjointness in the sample space
  - If $A \cap B = \emptyset$ and $P(A), P(B) > 0$, then $A$ and $B$ cannot be independent (e.g., $A$ and $A^c$)
- $A$ and $B$ are independent $\implies A^c$ and $B$ are independent
  - Because $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = P(A^c)P(B)$

# Independence example: Two rolls of a fair die (1/3)

## Example (2 successive die rolls)

Roll a fair die twice; all $6 \times 6 = 36$ outcomes are equally likely

**Question:** Are the following two events independent?

$$A_1 = \{\text{1st roll results in 6}\}, \qquad B_1 = \{\text{2nd roll results in 6}\}$$

**Answer:** Yes, because

$$P(A_1 \cap B_1) = P\big(\text{the outcome of the two rolls is } (6,6)\big) = \frac{1}{36}$$

$$P(A_1) = \frac{P\big(\{(6,1),(6,2),\ldots,(6,6)\}\big)}{P\big(\{(1,1),(1,2),\ldots,(6,6)\}\big)} = \frac{6}{36} = \frac{1}{6}$$

$$P(B_1) = \frac{P\big(\{(1,6),(2,6),\ldots,(6,6)\}\big)}{P\big(\{(1,1),(1,2),\ldots,(6,6)\}\big)} = \frac{6}{36} = \frac{1}{6}$$

Observe $P(A_1 \cap B_1) = P(A_1)\,P(B_1)$, which verifies the independence of $A_1$ and $B_1$)

# Independence example: Two rolls of a fair die (2/3)

## Example (2 successive die rolls)

Roll a fair die twice; all $6 \times 6 = 36$ outcomes are equally likely

**Question:** Are the following two events independent?

$$A_2 = \{\text{1st roll is 6}\}, \qquad B_2 = \{\text{the sum of the two rolls is 7}\}$$

**Answer:** Yes, because

$$P(A_2 \cap B_2) = P\big(\{(6,1)\}\big) = \frac{1}{36}$$

$$P(A_2) = \frac{P\big(\{(6,1),(6,2),\ldots,(6,6)\}\big)}{P\big(\{(1,1),(1,2),\ldots,(6,6)\}\big)} = \frac{6}{36} = \frac{1}{6}$$

$$P(B_2) = \frac{P\big(\{(6,1),(5,2),\ldots,(1,6)\}\big)}{P\big(\{(1,1),(1,2),\ldots,(6,6)\}\big)} = \frac{6}{36} = \frac{1}{6}$$

Observe $P(A_2 \cap B_2) = P(A_2)\,P(B_2)$, which verifies the independence of $A_2$ and $B_2$)

**Follow-up:** What about $B_2' = \{\text{the sum is 5}\}$? No, becauase $P(A_2 \cap B_2') = 0$ but $P(B_2') > 0$

# Independence example: Two rolls of a fair die (3/3)

## Example (2 successive die rolls)

Roll a fair die twice; all $6 \times 6 = 36$ outcomes are equally likely

**Question:** Are the following two events independent?

$$A_3 = \{\text{maximum of the two rolls is } 3\}, \qquad B_3 = \{\text{minimum of the two rolls is } 3\}$$

**Answer:** No, because

$$P(A_3 \cap B_3) = P\big(\{(3,3)\}\big) = \frac{1}{36}$$

$$P(A_3) = \frac{P\big(\{(1,3),(2,3),(3,3),(3,2),(3,1)\}\big)}{P\big(\{(1,1),(1,2),\ldots,(6,6)\}\big)} = \frac{5}{36}$$

$$P(B_3) = \frac{P\big(\{(3,6),(3,5),(3,4),(3,3),(4,3),(5,3),(6,3)\}\big)}{P\big(\{(1,1),(1,2),\ldots,(6,6)\}\big)} = \frac{7}{36}$$

Observe $P(A_3 \cap B_3) = \frac{1}{36} \neq \frac{35}{36^2} = P(A_3)\,P(B_3)$, thus $A_3$ and $B_3$) are not independent

# Conditional independence

Recall: for any event $C$ with $P(C) > 0$, the map $A \mapsto P(A \mid C)$ is itself a probability law

- So we can ask whether $A$ and $B$ are independent under this conditional law

**Definition (Conditional independence)**

The events $A$ and $B$ are **conditionally independent** given an event $C$ if

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

If $P(B \cap C) > 0$, this is equivalent to $P(A \mid B \cap C) = P(A \mid C)$ because

$$\begin{aligned} P(A \cap B \mid C) &= \frac{P(A \cap B \cap C)}{P(C)} && \text{by definition} \\ &= \frac{P(C)\, P(B \mid C)\, P(A \mid B \cap C)}{P(C)} && \text{by multiplicative rule} \\ &= P(B \mid C)\, P(A \mid B \cap C) \end{aligned}$$

- Once $C$ is known to have occurred, knowing $B$ gives no additional information about $A$
- Independence does not imply conditional independence, and vice versa

# Independence $\not\Longrightarrow$ conditional independence

**Example (2 fair coin tosses)**

Consider two independent fair coin tosses, and let

$$A = \{\text{1st toss is a head}\}$$
$$B = \{\text{2nd toss is a head}\}$$

- These two events are independent, but with

$$C = \{\text{the two tosses have different results}\}$$

- However, $A$ and $B$ are not conditionally independent given $C$ because

$$P(A \mid C) = \frac{1}{2}, \qquad P(B \mid C) = \frac{1}{2}, \qquad P(A \cap B \mid C) = 0$$

**Why?** Given $C$, the only possible outcomes are $HT$ and $TH$, so if one toss is Head, the other must be Tail

The independence of $A$ and $B$ does not imply conditional independence

# Conditional independence $\not\Longrightarrow$ Independence

## Example (2 different biased coin tosses)

Suppose there are two coins, fair and biased (say, landing on head with prob 0.9), and we choose one of the two at random with probability 1/2 each, and toss the coin twice:

$$A_i = \{\text{the i-th toss is a head}\}$$
$$B = \{\text{the coin is biased}\}$$

- The events $A_1$ and $A_2$ are independent, given the choice of a coin (=conditioned on $B$)
- However, without conditioning on $B$, $A_1$ and $A_2$ are not independent:

$$P(A_i) = P(B)\,P(A_i \mid B) + P(B^c)\,P(A_i \mid B^c) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{9}{10} = \frac{7}{10}, \qquad i \in \{1,2\},$$
$$P(A_1 \cap A_2) = P(B)\,P(A_1 \cap A_2 \mid B) + P(B^c)\,P(A_1 \cap A_2 \mid B^c) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{9}{10}\cdot\frac{9}{10} = \frac{53}{100}$$

**Why?** The hidden variable "which coin was chosen" creates dependence between the two tosses

The conditional independence of $A$ and $B$ does not imply unconditional independence

# Pop-up quiz

Two fair coin tosses. Let

$$A = \{\text{1st toss is H}\}, \qquad B = \{\text{2nd toss is H}\}, \qquad C = \{\text{the two tosses are different}\}.$$

Which statement is correct?

**a)** $A$ and $B$ are not independent, but are conditionally independent given $C$.

**b)** $A$ and $B$ are independent, and are conditionally independent given $C$.

**c)** $A$ and $B$ are independent, but are not conditionally independent given $C$.

**d)** $A$ and $B$ are neither independent nor conditionally independent given $C$.

**Answer: C.**

$$P(A) = P(B) = \frac{1}{2}, \qquad P(A \cap B) = \frac{1}{4},$$

so $A$ and $B$ are independent. But given $C$, the only outcomes are $HT$ and $TH$, so

$$P(A \mid C) = P(B \mid C) = \frac{1}{2}, \qquad P(A \cap B \mid C) = 0.$$

**Follow-up:** Would the answer change if we condition on $C^c$ instead of $C$?

# Independence of several events

We can extend the definition of the events for more than two events

**Definition (Independence (mutual independence))**

The events $A_1, A_2, \ldots, A_n$ are **independent** if

$$P\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i), \qquad \text{for every subset } S \subseteq \{1, 2, \ldots, n\}$$

For the case of three events, $A_1, A_2, A_3$ are independent if and only if

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1)\, P(A_2), \\ P(A_1 \cap A_3) &= P(A_1)\, P(A_3), \\ P(A_2 \cap A_3) &= P(A_2)\, P(A_3), \\ P(A_1 \cap A_2 \cap A_3) &= P(A_1)\, P(A_2)\, P(A_3) \end{aligned}$$

The first three (pairwise independence) do not imply the fourth, and vice versa

# Pairwise independence $\not\Longrightarrow$ mutual independence

## Example (2 fair coin tosses)

Consider two independent fair coin tosses, and let

$$\begin{aligned} A_1 &= \{\text{1st toss is a head}\} \\ A_2 &= \{\text{2nd toss is a head}\} \\ B &= \{\text{the two tosses have different results}\} \end{aligned}$$

- These three events are pairwise independent:
  - $A_1$ and $A_2$ are independent by definition
  - $A_1$ and $B$ are independent because

$$P(B \mid A_1) = \frac{P(A_1 \cap B)}{P(A_1)} = \frac{P\big(\{(H,T)\}\big)}{P\big(\{(H,T),(H,H)\}\big)} = \frac{1}{2} = P(B)$$

- However,

$$P(A_1 \cap A_2 \cap B) = 0 \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = P(A_1)\, P(A_2)\, P(B)$$

  and therefore, these three events are not independent

# Three-way factorization $\not\Longrightarrow$ mutual independence

## Example (2 fair die rolls)

Consider two independent rolls of a fair six-sided die, and let

$$A = \{\text{1st roll is 1, 2, or 3}\}$$
$$B = \{\text{1st roll is 3, 4, or 5}\}$$
$$C = \{\text{the sum of the two rolls is 9}\} = \{(3,6),(4,5),(5,4),(6,3)\}$$

- Observe that
$$P(A \cap B \cap C) = P\big(\{(3,6)\}\big) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4}{36} = P(A)\, P(B)\, P(C)$$
- Nevertheless, any pair of these three events are NOT independent:
$$P(A \cap B) = P\big(\{\text{1st roll is 3}\}\big) = \frac{1}{6} \neq P(A)\, P(B)$$
$$P(A \cap C) = P\big(\{(3,6)\}\big) = \frac{1}{36} \neq P(A)\, P(C)$$
$$P(B \cap C) = P\big(\{(3,6),(4,5),(5,4)\}\big) = \frac{1}{12} \neq P(B)\, P(C)$$

  and thus, these three events are not independent

# Wrap-up

## Independence

- $A$ and $B$ are independent if and only if $P(A \cap B) = P(A)P(B)$
- Independence means that learning one event does not change the probability of the other
- Independence is not the same as disjointness
- Independence of several events require factorization of probabilities over every subcollection

## Conditional independence

- $A$ and $B$ are conditionally independent given $C$ if and only if

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C).$$

- Independence does not imply conditional independence, and vice versa
- A hidden variable can create dependence marginally and independence after conditioning

*Suggested reading:* [BT08, Ch. 1.5]

# References

Dimitri Bertsekas and John N Tsitsiklis.
*Introduction to probability*, volume 1.
Athena Scientific, 2nd edition, 2008.