STA 131A: Introduction to Probability Theory

Lecture 5: Counting

Dogyoon Song

Spring 2026, UC Davis

Agenda

Last time: Independence

- Independence of two events
- Conditional independence
- Independence of several events

Today: Counting

- The counting principle
- Permutations, combinations, and partitions

Recap: Independence

The event A is **independent** of B if

$$P(A \cap B) = P(A)\, P(B)$$

- If $P(B) > 0$, this is equivalent to $P(A \mid B) = P(A)$
- Symmetric definition $\longrightarrow$ We can unambiguously say "*the events A and B are independent*"
- Independence is different from disjointness

The events A and B are **conditionally independent** given an event C if

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

- If $P(B \cap C) > 0$, this is equivalent to $P(A \mid C) = P(A \mid B \cap C)$
- Independence and conditional independence are different notions

The collection of events $A_1, A_2, \ldots, A_n$ are **(mutually) independent** if

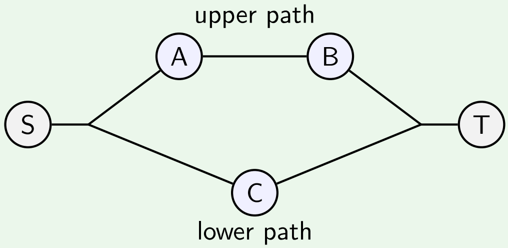
$$P\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i), \qquad \text{for } \textbf{every} \text{ subset } S \subseteq \{1, 2, \ldots, n\}$$

- Each subset S of indices imposes a distinct constraint; checking only pairs is not enough

Example: Using independence for modeling (1/2)

Example (Network connectivity)

Consider a network connecting two nodes S and T through intermediate nodes A, B, C as follows:



Suppose the intermediate nodes A, B, C fail independently, with probabilities

$$p_a = 0.1, \qquad p_b = 0.05, \qquad p_c = 0.2.$$

Question: What is the probability that there is a path connecting S and T without failure?

Example: Using independence for modeling (2/2)

Example (Network connectivity)

Answer: Define two events

$$U = \{\text{the upper path works}\}, \qquad L = \{\text{the lower path works}\}.$$

Then, since A, B are independent,

$$P(U) = (1 - p_a)(1 - p_b), \qquad P(L) = 1 - p_c.$$

Since U depends only on A, B and L depends only on C, the events U and L are independent. Thus,

$$\begin{aligned} P(\text{there is a working path}) &= P(U \cup L) \\ &= 1 - P(U^c \cap L^c) \\ &= 1 - (1 - P(U))(1 - P(L)) \end{aligned}$$

With $p_a = 0.1$, $p_b = 0.05$, $p_c = 0.2$,

$$P(U) = 0.9 \cdot 0.95 = 0.855, \qquad P(L) = 0.8,$$

and consequently,

$$P(\text{there is a working path}) = 1 - (1 - 0.855)(1 - 0.8) = 0.971$$

Example: Independent trials

Example (3 independent Bernoulli trials)

Consider a sequence of 3 independent Bernoulli trials (e.g., coin tosses):

$$P(\{\text{Head}\}) = p, \qquad P(\{\text{Tail}\}) = 1 - p$$

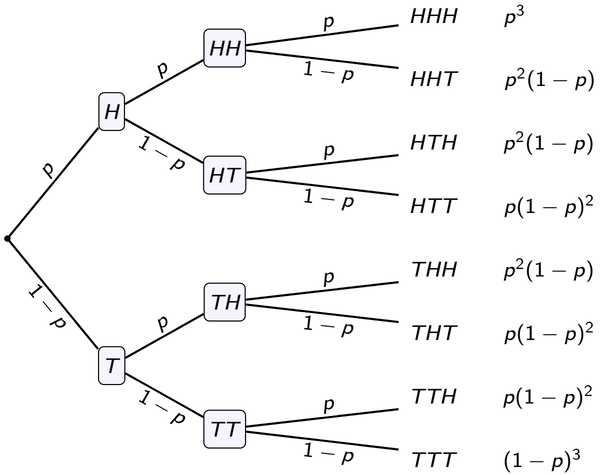
Question: Compute $p(k) = P(k \text{ heads})$

Answer: We can visualize independent Bernoulli trials using a sequential description (next slide)

$$p(k) = \binom{3}{k} p^k (1-p)^{3-k} = \begin{cases} p^3, & k = 3, \\ 3p^2(1-p), & k = 2, \\ 3p(1-p)^2, & k = 1, \\ (1-p)^3, & k = 0. \end{cases}$$

To compute $p(k)$, we need to count how many outcomes produce exactly k Heads
$\longrightarrow$ This leads naturally to the basic principles of counting

Illustration of 3 independent Bernoulli trials



Question: How can we count outcomes systematically, *without listing every outcome*?

The counting principle

For a finite sample space Ω with equally likely outcomes,

$$P(A) = \frac{|A|}{|\Omega|}$$

Question: How do we compute the number of elements in A and Ω?

The counting principle (product rule)

Suppose a procedure consists of r successive stages, and the following hold:

(a) There are n_1 possible results at the first stage

(b) For any sequence of possible results until stage $(i-1)$, there are n_i possible results at the i-th stage

Then the total number of possible results of this r-stage process is

$$n_1 n_2 \cdots n_r.$$

- **Divide-and-conquer:** Break a complicated counting problem into simple stages
- Multiply the number of choices across stages
- Example: 3 Bernoulli trials have $2 \cdot 2 \cdot 2 = 2^3$ possible outcomes

The counting principle: Examples

Example (The number of subsets)

Question: How many subsets does a set $A = \{a_1, a_2, \ldots, a_n\}$ have?

Answer: We can consider a sequential process that examines one element at a time to decide whether to include it in the subset or not. Therefore, the number of subsets is

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$$

Example (The number of telephone numbers)

A local telephone number consists of a 7-digit sequence whose first digit cannot be 0 or 1

Question: How many distinct telephone numbers are there within the local area?

Answer: We have a total of 7 stages, and a choice of one out of 10 elements at each stage, except for the first where we only have 8 choices. Therefore, the answer is

$$8 \times \underbrace{10 \times 10 \times \cdots \times 10}_{6 \text{ times}} = 8 \times 10^6$$

Permutations

A k-**permutation** of n distinct objects is an *ordered* selection of k of them

- The **factorial** of a nonnegative integer:

$$n! = n(n-1)\cdots 2\cdot 1, \qquad \text{with convention} \qquad 0! := 1$$

- The number of k-**permutations** is

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

  - As a special case, the number of permutations of n distinct objects is

$$n!$$

- **Example:** Choosing 2 objects from 4 in order gives

$$4\cdot 3 = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 1} = 12$$

Permutations: Examples

Example (Word of four distinct letters)

Question: How many 4-letter strings can be formed using distinct letters from the alphabet (A to Z)?

Answer: The total number of possible words is

$$\frac{n!}{(n-k)!} = \frac{26!}{22!} = 26 \cdot 25 \cdot 24 \cdot 23 = 358,800$$

Example (CDs of three types)

Suppose you have n_1 classical CDs, n_2 rock CDs, and n_3 country CDs (all distinct)

Question: In how many different ways can you arrange them so that CDs of the same type are contiguous?

Answer: Count in two stages:

- Choose the order of the three blocks: $3!$ ways
- Arrange the CDs within each block: $n_1!$, $n_2!$, and $n_3!$ ways

Therefore,

$$\#\{\text{valid arrangements}\} = 3!\, n_1!\, n_2!\, n_3!$$

Pop-up quiz

Question: How many different ways can we select a president and a vice president from 6 students?

a) $6 + 5$

b) $6 \cdot 5$

c) $6!$

d) $\binom{6}{2} := \frac{6!}{2!(6-2)!}$

Answer: B. There are 6 choices for president and then 5 remaining choices for vice president, so the answer is

$$6 \cdot 5 = 30$$

Follow-up: How many different choices of 2-person committees are there?
(Note: the order does not matter here)

Combinations

A **combination** is a selection in which order does *not* matter

- A k-**combination** of n distinct objects is a k-element subset
- Start from ordered selections ($=$ permutations) and group together duplicate orderings
  - Divide by the $k!$ reorderings of the same subset
  - The number of possible combinations is equal to the **binomial coefficient**

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

- **Example:** The number of distinct 2-subsets of a 4-element set $\{A, B, C, D\}$ is

$$\binom{4}{2} = 6$$

  - The 2-element subsets of $\{A, B, C, D\}$ are

$$AB,\ AC,\ AD,\ BC,\ BD,\ CD$$

Combinations: Example

Example (Exactly k Heads)

Question: In n independent Bernoulli trials, how many outcomes contain exactly k Heads?

Key observaiton: A sequence with exactly k Heads is determined by the k positions of the Heads

Therefore,

$$\#\{\text{outcome sequences with exactly } k \text{ Heads}\} = \binom{n}{k}$$

- For example, with $n = 3$ and $k = 2$, there are $\binom{3}{2} = 3$ sequences:

$$HHT, \qquad HTH, \qquad THH$$

Consequently, if $P(\text{Head}) = p$, then

$$P(k \text{ Heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

Partitions

Suppose n distinct objects are to be divided into r *labeled* groups of sizes

$$n_1, n_2, \ldots, n_r, \qquad n_1 + \cdots + n_r = n$$

The number of such partitions gives the **multinomial coefficient**

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \, \cdots \, n_r!}$$

Two alternative viewpoints to obtain the formula:

- **Successive-choice view:**

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r}$$

- **Permutation view:** Start with $n!$ orders and divide by the $n_i!$ reorderings inside each group

Partitions: Example

Example (Anagrams of TATTOO)

Question: How many different letter sequences can be obtained by rearranging the 6 letters in the word TATTOO?

Key observaiton: There are 6 positions, and we must choose positions for

3 T's, 2 O's, 1 A

That is, we are partitioning the 6 positions into groups of sizes $3, 2, 1$

Thus, the number of distinct words is

$$\binom{6}{3,2,1} = \frac{6!}{3!\,2!\,1!} = 60$$

Alternative, yet equivalent approach: Start from 6! arrangements of labeled letters $T_1, T_2, T_3, O_1, O_2, A$ and divide by 3! for the repeated T's and by 2! for the repeated O's

Wrap-up

Counting principle

- Break a counting problem into stages and multiply the number of choices across stages
- For equally likely outcomes, probabilities reduce to counting: $P(A) = \frac{|A|}{|\Omega|}$

Permutations, combinations and partitions

- Permutations: order matters

$$\frac{n!}{(n-k)!}$$

- Combinations: order does not matter

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Partitions: Dividing n distinct objects into labeled groups of sizes $n_1, \ldots, n_r$ gives

$$\binom{n}{n_1, \ldots, n_r} = \frac{n!}{n_1! \cdots n_r!}$$

Suggested reading: [BT08, Ch. 1.6]

References

Dimitri Bertsekas and John N Tsitsiklis.
Introduction to probability, volume 1.
Athena Scientific, 2nd edition, 2008.