STA 131A: Introduction to Probability Theory

Lecture 6: Discrete Random Variables

Dogyoon Song

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Recap: Counting

In a finite sample space with equally likely outcomes,

$$P(A) = \frac{|A|}{|\Omega|}.$$

So many probability problems reduce to counting

The counting principle supports a divide-and-conquer approach

Example: Among 100 students enrolled in STA 131A,

- An ordered list of 10 students to call $\rightarrow$ permutation
- A 10-student committee $\rightarrow$ combination
- An assignment of the 100 students to 10 labeled groups of size 10 $\rightarrow$ partition

Agenda

Last time: Counting

- The counting principle
- Permutations, combinations, and partitions

Today:

- Discrete random variables
- Probability mass function
- Common discrete models: Bernoulli, binomial, geometric, Poisson

Motivation: From outcomes to numbers

In many probabilistic models, the outcomes are numerical

- Sometimes outcomes themselves are numeric
- Even if not, we care about some numbers related to outcomes

A random variable X extracts a numerical feature of the outcome

- 3 coin tosses $\rightarrow$ number of Heads
- 2 die rolls $\rightarrow$ sum of the faces
- game outcome $\rightarrow$ monetary payoff

Once outcomes are mapped to numbers, we can ask questions such as

- What is the probability of $X = 2$?
- What is the probability of $X \geq 3$?
- What is the "typical" value of X?

Random variable: Definition

Definition (Random variable)

A **random variable** is a function $X : \Omega \to \mathbb{R}$

- A random variable assigns a number to each outcome $\omega \in \Omega$
- The randomness comes from the outcome ω, not from the function X itself
- X is **discrete** if its range is finite or countably infinite

Examples:

- *Discrete:* number of Heads in n coin tosses; number of trials until the first Head
- *Continuous:* landing location of a dart on a line segment

Notation: For simplicity, we may write

$$P(X = x) = P(\{\omega \in \Omega : X(\omega) = x\}), \qquad P(X \in S) = P(\{\omega \in \Omega : X(\omega) \in S\})$$

Probability mass function

Definition (Probability mass function)

If X is a discrete random variable, its **probability mass function (PMF)** is a function $p_X : \mathbb{R} \to \mathbb{R}$ such that

$$p_X(x) = P(X = x), \qquad x \in \mathbb{R}$$

- $p_X(x) \geq 0$ for all x
- $\sum_{x \in \mathcal{X}} p_X(x) = 1$, where $\mathcal{X} = \{x : p_X(x) > 0\}$
- For any set $S \subseteq \mathbb{R}$,

$$P(X \in S) = \sum_{x \in S} p_X(x)$$

The PMF completely determines the distribution of a discrete random variable

The **support** of X is

$$\mathcal{X} = \{x \in \mathbb{R} : p_X(x) > 0\}.$$

Probability mass function: Example

Example (2 fair coin tosses)

Sample space: $\Omega = \{HH, HT, TH, TT\}$

Define a random variable X denoting the number of Heads:

$$X(HH) = 2, \qquad X(HT) = 1, \qquad X(TH) = 1, \qquad X(TT) = 0$$

The support is $\{0, 1, 2\}$, and because the four outcomes are equally likely,

$$P(X = 2) = 1/4, \qquad P(X = 1) = 1/2, \qquad P(X = 0) = 1/4$$

Thus, the PMF of X is

$$p_X(x) = \begin{cases} 1/4, & x = 0, \\ 1/2, & x = 1, \\ 1/4, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Pop-up quiz

For two fair coin tosses, let X be the number of Heads.

Question: Which statement is correct?

A) $p_X(HT) = 1/4$

B) $\{X = 1\}$ is a random variable

C) $P(X \in \{0, 2\}) = 1/2$

D) The support of X is $\{HH, HT, TH, TT\}$

Answer: C.
$P(X = 0) + P(X = 2) = 1/4 + 1/4 = 1/2$; $\{X = 1\}$ is an event, and the support of X is $\{0, 1, 2\}$.

Follow-up: How would you compute $P(X \geq 1)$ directly from the PMF?

Bernoulli random variable

One success/failure trial

Notation:

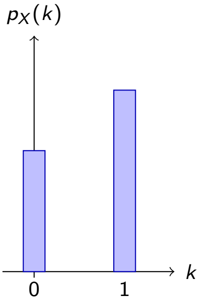
$$X \sim \mathrm{Bernoulli}(p), \qquad 0 \le p \le 1.$$

PMF:

$$p_X(k) = \begin{cases} p, & k = 1, \\ 1 - p, & k = 0. \end{cases}$$

Support: $\{0, 1\}$

Example PMF: $p = 0.6$



- $X = 1$ means success, $X = 0$ means failure
- Example: one coin toss; one email is spam or not spam
- Bernoulli variables are the building blocks for many common probabilistic models

Binomial random variable

Number of successes in n independent Bernoulli(p) trials

Notation:

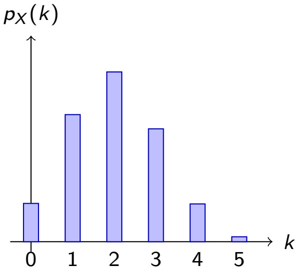
$$X \sim \mathrm{Binomial}(n, p).$$

PMF:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

Support: $\{0, 1, \ldots, n\}$

Example PMF: $n = 5$, $p = 0.4$



- X counts how many successes occur in a fixed number of trials
- Connection to Bernoulli: $X = X_1 + \cdots + X_n$, where X_i are independent Bernoulli(p)
- Example: number of Heads in n coin tosses

Geometric random variable

Waiting time until the first success in repeated independent Bernoulli(p) trials

Notation:

$$X \sim \mathrm{Geometric}(p).$$

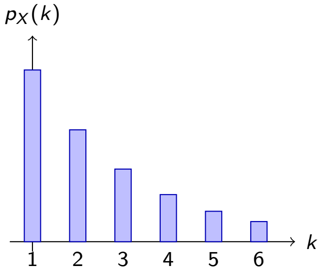
PMF:

$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1, 2, 3, \ldots$$

Support: $\{1, 2, 3, \ldots\}$

- $X = k$ means the first success occurs on trial k
- Connection to Bernoulli: the trials are i.i.d. Bernoulli(p); we stop at the first success
- Example: number of tosses until the first Head

Example PMF: $p = 0.35$



Poisson random variable

Number of occurrences in a fixed time or space window

Notation:

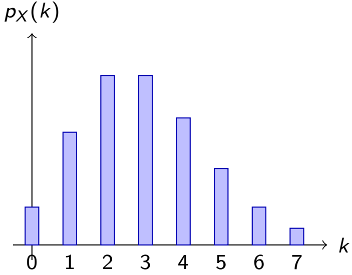
$$X \sim \mathrm{Poisson}(\lambda), \qquad \lambda > 0.$$

PMF:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

Support: $\{0, 1, 2, \ldots\}$

Example PMF: $\lambda = 3$



- X counts how many times an event occurs in a fixed window
- Connection to Bernoulli: it can arise as a limit of many rare Bernoulli opportunities, with $np \approx \lambda$
- Example: number of emails in an hour; number of typos on a page.

Pop-up quiz

Suppose independent Bernoulli(p) trials are repeated until the first success occurs

Question: Which model best describes the random variable

$$X = \text{the trial on which the first success occurs?}$$

A) Bernoulli(p)
B) Binomial(n, p)
C) Geometric(p)
D) Poisson(λ)

Answer: C.
Because X is a waiting time to the first success in repeated independent Bernoulli trials.

Follow-up: If we instead run exactly n trials and count the total number of successes, which model do we get?

Wrap-up

Random variables

- A random variable is a function $X : \Omega \to \mathbb{R}$ that assigns a number to each outcome
- Discrete random variables take finitely many or countably many values

Probability mass functions

- The PMF is $p_X(x) = P(X = x)$
- For a discrete random variable, $p_X(x) \geq 0$ and $\sum_x p_X(x) = 1$
- For any set $S \subseteq \mathbb{R}$, $P(X \in S) = \sum_{x \in S \cap \mathcal{X}} p_X(x)$

Common discrete models

- Bernoulli: one success/failure trial
- Binomial: number of successes in n independent Bernoulli trials
- Geometric: number of trials until the first success
- Poisson: count of occurrences in a fixed window

Suggested reading: [BT08, Ch. 2.1–2.2]

References

Dimitri Bertsekas and John N Tsitsiklis.
Introduction to probability, volume 1.
Athena Scientific, 2nd edition, 2008.