STA 131A: Introduction to Probability Theory

Lecture 14: Normal Random Variables

Dogyoon Song

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Announcements

Mid-course survey (Due: tonight)

- Please take 10 minutes to complete the survey on Canvas
- Concrete feedback and constructive suggestions will be appreciated

Remote lecture next Monday (May 4): More details will be announced on Canvas

Agenda

Last time: CDFs

- $F_X(x) = P(X \leq x)$
- $P(a < X \leq b) = F_X(b) - F_X(a)$
- Discrete mass = jumps; continuous density = slope

Today: Normal random variables

- Normal PDF and its parameters
- Standard normal distribution and the normal table
- Standardization
- Worked examples

Recap: Cumulative distribution functions

For any random variable X,

$$F_X(x) = P(X \leq x).$$

- $P(a < X \leq b) = F_X(b) - F_X(a)$

For a discrete random variable,

$$F_X(x) = \sum_{x' \leq x} p_X(x'), \qquad p_X(x_i) = F_X(x_i) - F_X(x_i^-),$$

where $F_X(x_i^-) := \lim_{x' \to x_i -} F_X(x')$ is the value just before the jump.

For a continuous random variable,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt, \qquad f_X(x) = F_X'(x)$$

when F_X is differentiable.

Normal random variable

Definition (Normal random variable)

A random variable X is **normal** with mean μ and variance σ^2, written

$$X \sim N(\mu, \sigma^2),$$

if its PDF is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \qquad x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

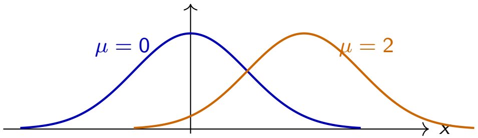
- $\mathbb{E}[X] = \mu$: location/center
- $\mathrm{Var}(X) = \sigma^2$, so σ is the standard deviation/spread parameter

Normal PDF: Shape and parameters

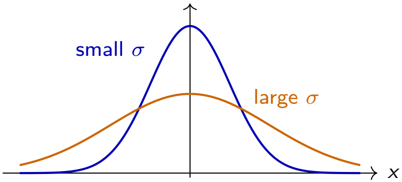
Properties of the normal PDF

- Symmetric around μ
- Bell-shaped
- Total area under the curve is 1
- μ controls location; σ controls spread

Changing μ: shift



Changing σ: spread



Normality preserved under linear transformations

If $X \sim N(\mu, \sigma^2)$, then for constants a, b with $a \neq 0$,

$$aX + b \sim N(a\mu + b,\ a^2\sigma^2).$$

Thus:

$$\mathbb{E}[aX + b] = a\mu + b, \qquad \mathrm{Var}(aX + b) = a^2\sigma^2.$$

Why is this true? One can verify it by deriving the PDF of $aX + b$.

- We will develop the general transformation method later

Key consequence: every normal random variable can be converted into a standard normal random variable.

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

Standard normal random variable

Definition (Standard normal)

The **standard normal** random variable is

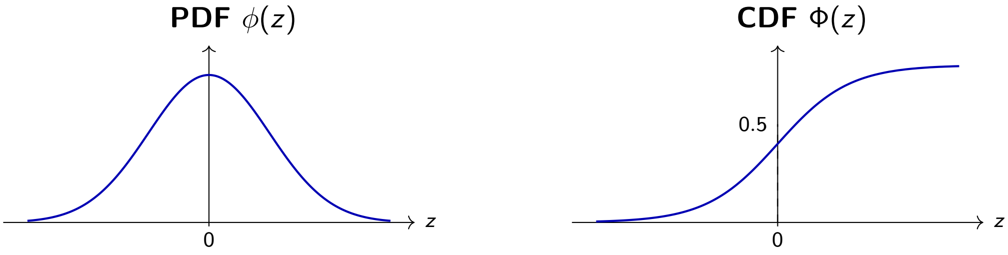
$$Z \sim N(0, 1).$$

Its PDF and CDF are respectively denoted by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad \Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$$

There is no simple elementary formula for $\Phi(z)$, so we use a standard normal table, calculator, or software.

Standard normal PDF and CDF: Visual summary



By symmetry,

$$\Phi(-z) = 1 - \Phi(z).$$

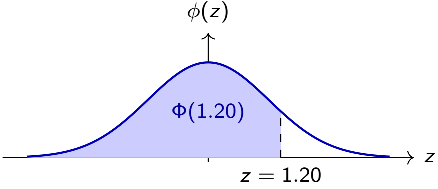
Useful consequences:

$$\Phi(0) = \frac{1}{2}, \qquad P(Z > z) = 1 - \Phi(z), \qquad P(|Z| \leq z) = 2\Phi(z) - 1 \quad (z \geq 0).$$

Standard normal table

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574

What the table gives



Example table entry

$$\Phi(1.20) = P(Z \leq 1.20) \approx 0.8849.$$

z	1.18	1.19	1.20
$\Phi(z)$	0.8810	0.8830	0.8849

Standardization

If $X \sim N(\mu, \sigma^2)$, then we can "standardize" it by defining

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

Observe that for any $x \in \mathbb{R}$,

$$X \leq x \qquad \Longleftrightarrow \qquad Z \leq \frac{x - \mu}{\sigma}$$

Therefore,

$$\begin{aligned} P(a < X \leq b) &= P\left(\frac{a - \mu}{\sigma} < \frac{X - \mu}{\sigma} \leq \frac{b - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right). \end{aligned}$$

One normal table is enough for all normal distributions.

Using the standard normal table

Commonly used identities:

$$P(Z \le z) = \Phi(z),$$
$$P(Z > z) = 1 - \Phi(z), \qquad \because \Phi(-z) = 1 - \Phi(z)$$
$$P(a < Z \le b) = \Phi(b) - \Phi(a).$$

Typical workflow:

1. Convert X to $Z = (X - \mu)/\sigma$.
2. Rewrite the probability in terms of Z.
3. Use the table for Φ.

Remark: Most tables give left-tail probabilities $\Phi(z) = P(Z \le z)$. For right tails or middle intervals, first rewrite the probability in terms of Φ.

Example: Normal-table calculation

Example

Suppose

$$X \sim N(70, 10^2).$$

Compute $P(60 \leq X \leq 85)$ (for a continuous random variable, endpoints do not matter).

Standardize:

$$P(60 \leq X \leq 85) = P\left(\frac{60-70}{10} \leq Z \leq \frac{85-70}{10}\right) = P(-1 \leq Z \leq 1.5).$$

Thus,

$$P(-1 \leq Z \leq 1.5) = \Phi(1.5) - \Phi(-1).$$

Using $\Phi(1.5) \approx 0.9332$ and $\Phi(-1) \approx 0.1587$,

$$P(60 \leq X \leq 85) \approx 0.7745.$$

Example: Tail probability

Example

Let

$$X \sim N(100, 15^2).$$

Compute $P(X > 130)$.

$$P(X > 130) = P\left(Z > \frac{130 - 100}{15}\right) = P(Z > 2).$$

Thus,

$$P(Z > 2) = 1 - \Phi(2).$$

Using $\Phi(2) \approx 0.9772$,

$$P(X > 130) \approx 0.0228.$$

Interpretation: A value 2 standard deviations above the mean is relatively rare.

Pop-up quiz

Suppose $X \sim N(10, 4)$.

Question: Which expression equals $P(X \leq 13)$?

A) $\Phi(13)$
B) $\Phi(1.5)$
C) $\Phi(0.75)$
D) $1 - \Phi(1.5)$

Answer: B.
Since the standard deviation is 2, we standardize and obtain

$$\frac{13 - 10}{2} = 1.5,$$

so

$$P(X \leq 13) = P(Z \leq 1.5) = \Phi(1.5).$$

Follow-up: What expression gives $P(X > 13)$?

Pop-up quiz

Let $Z \sim N(0,1)$. Suppose $\Phi(1.2) = 0.8849$.

Question: Which expression equals $P(-1.2 \leq Z \leq 1.2)$?

- **A)** 0.8849
- **B)** $1 - 0.8849$
- **C)** $2(0.8849) - 1$
- **D)** $1 - 2(0.8849)$

Answer: C.
By symmetry, $\Phi(-1.2) = 1 - \Phi(1.2)$. Hence

$$P(-1.2 \leq Z \leq 1.2) = \Phi(1.2) - \Phi(-1.2) = 2\Phi(1.2) - 1.$$

Follow-up: Express $P(Z > 1.2)$ using $\Phi(1.2)$.

Additional exercise: Manufacturing tolerance (1/2)

Example (A more involved normal calculation)

A machine produces rods whose lengths are approximately normal:

$$X \sim N(10,\ 0.2^2)$$

in centimeters.
A rod is acceptable if

$$9.7 \leq X \leq 10.3.$$

Questions:

1. What fraction of rods are acceptable?
2. Among 100 independently produced rods, what is the expected number acceptable?
3. Keeping the mean at 10, how should we change the standard deviation if we want this fraction to increase?

Additional exercise: Manufacturing tolerance (2/2)

Example

Standardize X:

$$P(9.7 \le X \le 10.3) = P\left(\frac{9.7 - 10}{0.2} \le Z \le \frac{10.3 - 10}{0.2}\right) = P(-1.5 \le Z \le 1.5) = \Phi(1.5) - \Phi(-1.5).$$

Using symmetry, $\Phi(-1.5) = 1 - \Phi(1.5)$, and therefore,

$$P(-1.5 \le Z \le 1.5) = 2\Phi(1.5) - 1 \approx 2 \times 0.933 - 1 = 0.866.$$

If N is the number acceptable among 100 independently produced rods, then

$$N \sim \mathrm{Binomial}(100, 0.866),$$

where 0.866 is the acceptance probability for one rod. Hence

$$\mathbb{E}[N] = 100 \times 0.866) = 86.6.$$

Reducing σ concentrates the distribution around the mean and increases the acceptable fraction.

Wrap-up

Normal random variable: $X \sim N(\mu, \sigma^2)$ has mean μ and variance σ^2

- The normal PDF is symmetric and bell-shaped.
- μ controls location; σ controls spread.

Standardization: If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1), \qquad \text{and} \qquad P(a < X \leq b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right).$$

What you should practice:

- Translate every normal probability into a standard normal probability.
- Use complements and symmetry to convert table values into the desired probability.
- Be careful about variance vs. standard deviation and left-tail vs. right-tail probabilities.

Suggested reading: [BT08, Ch. 3.3]

References

Dimitri Bertsekas and John N Tsitsiklis.
Introduction to probability, volume 1.
Athena Scientific, 2nd edition, 2008.