STA 131A: Introduction to Probability Theory

Lecture 15: Joint PDFs of Multiple Random Variables

Dogyoon Song

Spring 2026, UC Davis

Agenda

Last time:

- Continuous RVs and PDFs
- Cumulative distribution functions (CDFs)
- Normal distribution and standardization

Today: joint PDFs

- Joint PDFs of two continuous random variables
- Marginal PDFs
- Joint CDFs
- Expectations with joint PDFs
- Extension to more than two random variables

Recap: One continuous random variable

A random variable X is continuous with PDF f_X if

$$P(X \in B) = \int_B f_X(x)\,dx.$$

- A valid PDF satisfies

$$f_X(x) \geq 0, \qquad \text{and} \qquad \int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

- For intervals:

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx.$$

- Expectation:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx.$$

The cumulative distribution function (CDF) of X:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt.$$

Jointly continuous random variables

Definition (Joint PDF)

Random variables X and Y are **jointly continuous** if there exists a nonnegative function $f_{X,Y}$ such that

$$P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\, dx\, dy$$

for regions $A \subseteq \mathbb{R}^2$. The function $f_{X,Y}$ is called the **joint PDF** of X and Y.

A valid joint PDF satisfies

$$f_{X,Y}(x,y) \geq 0, \qquad \text{and} \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1.$$

Joint probability as volume under a surface

For jointly continuous X, Y,

$$P((X, Y) \in A) = \iint_A f_{X,Y}(x, y)\, dx\, dy.$$

Interpretation:

- The joint PDF is a density surface over the (x, y)-plane.
- Probability is volume under the surface over the region A.
- A value $f_{X,Y}(x, y)$ is not itself a probability.

For rectangles:

$$P(a \leq X \leq b,\ c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y)\, dy\, dx.$$

- For jointly continuous variables, including/excluding boundary does not change the probability.

Not every continuous pair is jointly continuous

Even if X and Y are individually continuous, the pair (X, Y) need not be jointly continuous.

Example

Let X be continuous, for example $X \sim \mathrm{Uniform}(0, 1)$, and let

$$Y = X.$$

Then both X and Y are continuous marginally, but

$$P(Y = X) = 1.$$

A joint PDF over the plane cannot place positive probability on a single line, because a line has zero area.

Thus, (X, Y) is not jointly continuous, even though both marginal random variables are continuous.

Marginal PDFs

Given a joint PDF $f_{X,Y}$, the marginal PDFs are obtained by integrating out the other variable:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$$

This is the continuous analogue of summing out a variable from a joint PMF:

$$p_X(x) = \sum_y p_{X,Y}(x,y).$$

Key idea:

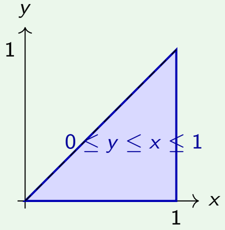
$$\text{discrete sums} \quad \longleftrightarrow \quad \text{continuous integrals.}$$

Example: Uniform density on a triangle (1/2)

Example

Let (X, Y) be uniformly distributed over the triangle

$$T = \{(x, y) : 0 \leq y \leq x \leq 1\}.$$



Thus,

$$f_{X,Y}(x, y) = \begin{cases} c, & \text{if } (x, y) \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Since the area of T is $1/2$, we need

$$c \cdot \frac{1}{2} = 1, \qquad \Longrightarrow \qquad c = 2.$$

Therefore,

$$f_{X,Y}(x, y) = \begin{cases} 2, & 0 \leq y \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example: Uniform density on a triangle (2/2)

Example

Next, we compute the marginal PDFs from the joint PDF

$$f_{X,Y}(x,y) = 2, \qquad 0 \le y \le x \le 1.$$

We find f_X by fixing x and integrating over the allowed values $0 \le y \le x$:

$$f_X(x) = \int_0^x 2\,dy = 2x, \qquad 0 \le x \le 1.$$

We find f_Y by fixing y. Since $0 \le y \le x \le 1$, the allowed values of x are $y \le x \le 1$:

$$f_Y(y) = \int_y^1 2\,dx = 2(1-y), \qquad 0 \le y \le 1.$$

Thus,

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases} \qquad f_Y(y) = \begin{cases} 2(1-y), & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Pop-up quiz

For the triangular density

$$f_{X,Y}(x,y) = 2, \qquad 0 \le y \le x \le 1,$$

which expression gives the marginal density $f_Y(y)$ for $0 \le y \le 1$?

A) $\int_0^y 2\,dx$

B) $\int_y^1 2\,dx$

C) $\int_0^x 2\,dy$

D) $2y$

Answer: B.

For fixed y, the allowed x-values run from $x = y$ to $x = 1$.

Follow-up: If instead we compute $f_X(x)$, which variable is fixed, and what are the allowed values of the other variable?

Example: Buffon's needle setup

Example (Buffon's needle)

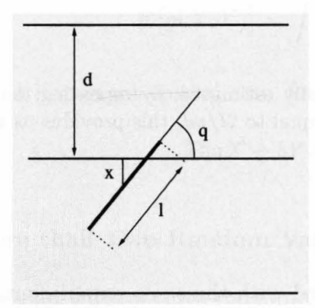
Parallel lines are spaced distance d apart. A needle of length $\ell \leq d$ is dropped randomly.

Let

$X =$ distance from the needle center to the nearest line,

and let

$\Theta =$ acute angle between the needle and the parallel lines.



Assume

$$X \sim \mathrm{Uniform}(0, d/2), \qquad \Theta \sim \mathrm{Uniform}(0, \pi/2),$$

and X, Θ are independent.

Buffon's needle probability

Example (Buffon's needle)

By uniformity and independence, the joint PDF of X and Θ is constant:

$$f_{X,\Theta}(x,\theta) = \frac{1}{(d/2)(\pi/2)} = \frac{4}{\pi d},$$

for

$$0 \le x \le d/2, \qquad 0 \le \theta \le \pi/2.$$

Since $\ell \le d$, we have $(\ell/2)\sin\theta \le d/2$, so the crossing probability is

$$\begin{aligned} P(\text{cross}) &= \int_0^{\pi/2} \int_0^{(\ell/2)\sin\theta} \frac{4}{\pi d}\, dx\, d\theta \\ &= \frac{4}{\pi d} \int_0^{\pi/2} \frac{\ell}{2} \sin\theta\, d\theta \\ &= \frac{2\ell}{\pi d}. \end{aligned}$$

Joint CDF

The **joint CDF** of X and Y is

$$F_{X,Y}(x,y) = P(X \le x,\ Y \le y).$$

For jointly continuous random variables,

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, dt\, ds.$$

When the derivatives exist,

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\, \partial y} F_{X,Y}(x,y).$$

Expectation with a joint PDF

For jointly continuous X, Y,

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy.$$

In particular,

$$\mathbb{E}[X] = \iint x f_{X,Y}(x, y)\, dx\, dy, \qquad \mathbb{E}[Y] = \iint y f_{X,Y}(x, y)\, dx\, dy.$$

Linearity of expectation still holds:

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c.$$

No independence is required for linearity of expectation.

Worked exercise: Triangle density

For the triangular density

$$f_{X,Y}(x,y) = 2, \qquad 0 \le y \le x \le 1,$$

compute $P(Y \le X/2)$, $\mathbb{E}[X]$, and $\mathbb{E}[Y]$.

Inside the triangle $0 \le y \le x \le 1$, the event $Y \le X/2$ corresponds to $0 \le y \le x/2$. Thus

$$P(Y \le X/2) = \int_0^1 \int_0^{x/2} 2\, dy\, dx = \int_0^1 x\, dx = \frac{1}{2}.$$

Also,

$$\mathbb{E}[X] = \int_0^1 \int_0^x 2x\, dy\, dx = \int_0^1 2x^2\, dx = \frac{2}{3},$$
$$\mathbb{E}[Y] = \int_0^1 \int_0^x 2y\, dy\, dx = \int_0^1 x^2\, dx = \frac{1}{3}.$$

Pop-up quiz

Suppose X, Y are jointly continuous with joint PDF $f_{X,Y}$. What is $P(X = Y)$?

A) Always 1

B) Always 0

C) Equal to $\int f_{X,Y}(x, x)\, dx$

D) Cannot be determined from the joint PDF

Answer: B.

The event $\{X = Y\}$ lies on a line in the plane, which has area zero under a joint PDF.

Follow-up: Why did the example $Y = X$ fail to be jointly continuous?

More than two continuous random variables

The joint PDF of $X_1, \ldots, X_n$ is a nonnegative function

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

such that

$$\int \cdots \int f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = 1.$$

Probabilities are computed by integrating over regions:

$$P((X_1, \ldots, X_n) \in A) = \int_A f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

Marginals are obtained by integrating out the variables not of interest.

Wrap-up

X, Y are **jointly continuous** with **joint PDF** $f_{X,Y}$ if

$$P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy.$$

- A joint PDF assigns probability through area/volume, not through point values.
- **Marginals** are obtained by integrating out the other variables:

$$f_X(x) = \int f_{X,Y}(x,y)\,dy.$$

- **Expectations** are weighted averages over the joint density:

$$\mathbb{E}[g(X,Y)] = \iint g(x,y) f_{X,Y}(x,y)\,dx\,dy.$$

Caution

- Continuous marginals do not guarantee joint continuity.
- Geometry matters: always identify the region of integration carefully.

Suggested reading: [BT08, Ch. 3.4]

References

Dimitri Bertsekas and John N Tsitsiklis.
Introduction to probability, volume 1.
Athena Scientific, 2nd edition, 2008.