

STA 131A: Introduction to Probability Theory

Lecture 18: Derived Distributions

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Announcement

Homework 5 is due tomorrow

Midterm 2 on Fri, May 15 (10:00 am–10:50 am in class)

- **Arrive early:** The exam starts at 10:00 am and ends at 10:50 am sharp
- **One hand-written cheat sheet:** Letter-size (8.5"×11"), double-sided, brief formulas/notes
- **Calculator:** A simple (non-graphing) scientific calculator is allowed
- **No other materials** beyond the single cheat sheet (no textbooks, etc.) is allowed
- **SDC accommodations:** Confirm scheduling with AES online ASAP

Advice for preparation:

- Primary coverage: Lectures 12–19
- Core concepts from earlier may be assumed
- 2 practice midterms and brief solution keys are posted on course webpage
- Office hours next week:
 - Instructor: Wed, 2:30–3:30 pm
 - TA: Tue 3:00–5:00 pm

Agenda

So far: Continuous random variables

- Continuous random variable and PDF
- Cumulative distribution function
- Normal random variable
- Jointly continuous random variables
- Conditioning

Today: Derived distributions

- Functions of one random variable: $Z = g(X)$
 - Generic method: support \rightarrow CDF \rightarrow PDF
 - Monotone transformation formula
 - Linear transformations as a special case
- Functions of two random variables: $Z = h(X, Y)$
 - CDF by integrating over a region
 - Sum of independent random variables: convolution

Recap: Continuous random variable

A random variable X is continuous if it has a PDF f_X :

$$P(X \in A) = \int_A f_X(x) dx.$$

Its CDF satisfies

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt, \quad f_X(x) = F'_X(x) \quad (\text{when differentiable}).$$

A pair of random variables X, Y is jointly continuous if $\exists f_{X,Y}$ such that

$$P((X, Y) \in A) = \iint_A f_{X,Y}(x, y) dx dy.$$

Question for today: If

$$Z = g(X) \quad \text{or} \quad Z = h(X, Y),$$

how do we find the support, CDF, and PDF of Z ?

Distribution of a function of a random variable

Let X be a random variable and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Let

$$Z = g(X).$$

- When X is discrete, we computed the PMF of Z by

$$p_Z(z) = \sum_{x:g(x)=z} p_X(x).$$

- When X is continuous, $P(X = x) = 0$ for all x ; can we sum $f_X(x)$ over x s.t. $g(x) = z$? **No!**

CDF-first workflow for $Z = g(X)$

1. Identify the support: possible values of $Z = g(X)$.
2. Compute the CDF:

$$F_Z(z) = P(Z \leq z) = P(g(X) \leq z).$$

3. Differentiate on the support:

$$f_Z(z) = \frac{d}{dz} F_Z(z).$$

Example: A quadratic transformation

Example

Let $X \sim \text{Uniform}(-1, 1)$, and let

$$Z = X^2.$$

Question: Find the PDF of Z .

- Observe that (1) for $z < 0$, $F_Z(z) = 0$; and (2) for $z \geq 1$, $F_Z(z) = 1$.
- For $0 \leq z < 1$,

$$F_Z(z) = P(Z \leq z) = P(X^2 \leq z) = P(-\sqrt{z} \leq X \leq \sqrt{z}) = \frac{2\sqrt{z}}{2} = \sqrt{z}.$$

Thus,

$$F_Z(z) = \begin{cases} 0, & z < 0, \\ \sqrt{z}, & 0 \leq z < 1, \\ 1, & z \geq 1, \end{cases} \quad \text{and therefore} \quad f_Z(z) = F'_Z(z) = \begin{cases} \frac{1}{2\sqrt{z}}, & 0 < z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Check: $\int_0^1 \frac{1}{2\sqrt{z}} dz = 1$.

Example: An exponential transformation

Example

Let $X \sim \text{Uniform}(0, 1)$, and let

$$Z = e^X.$$

Question: Find the PDF of Z .

Since $Z = e^X$, the possible values are $1 < Z < e$.

For $1 < z < e$,

$$F_Z(z) = P(Z \leq z) = P(e^X \leq z) = P(X \leq \log z) = \log z.$$

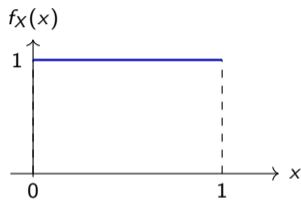
Thus,

$$F_Z(z) = \begin{cases} 0, & z \leq 1, \\ \log z, & 1 < z < e, \\ 1, & z \geq e, \end{cases} \quad \text{and therefore} \quad f_Z(z) = \begin{cases} \frac{d}{dz} \log z = \frac{1}{z}, & 1 < z < e, \\ 0, & \text{otherwise.} \end{cases}$$

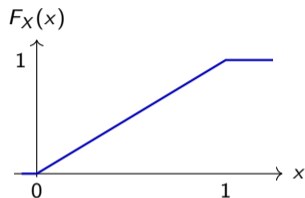
Observation: As $z = e^x$ stretches more for larger x , the density of Z decreases like $1/z$. 7/28

Visual illustration: $Z = e^X$

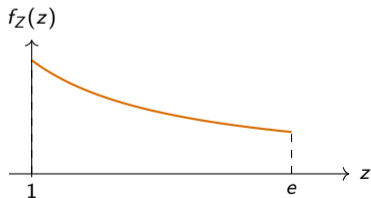
PDF of $X \sim \text{Uniform}(0, 1)$



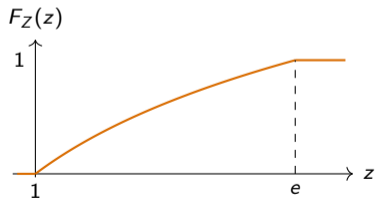
CDF of X



PDF of $Z = e^X$



CDF of $Z = e^X$



Message: A transformation may change both the support and the shape of the density.

PDF formula for a strictly monotone transformation

Definition (Strictly monotone on the support)

A function g is strictly monotone on the support of X if it is either strictly increasing or strictly decreasing there.

A strictly monotone function g is injective (one-to-one), so it has an inverse.

PDF formula for strictly monotonic function

Suppose X is continuous and $Z = g(X)$, where g is strictly monotone and differentiable on the support of X , with inverse

$$x = h(z) = g^{-1}(z).$$

Then, for z in the support of Z ,

$$f_Z(z) = f_X(h(z)) \cdot |h'(z)|.$$

Outside the support, $f_Z(z) = 0$.

Why the monotone formula works

Why does this formula hold? Assume g is strictly increasing, and let $h = g^{-1}$. Then

$$F_Z(z) = P(Z \leq z) = P(g(X) \leq z) = P(X \leq h(z)) = F_X(h(z)).$$

Differentiating with respect to z , we obtain

$$f_Z(z) = F'_Z(z) = f_X(h(z))h'(z).$$

If g is strictly decreasing, the inequality reverses:

$$F_Z(z) = P(X \geq h(z)) = 1 - F_X(h(z)),$$

where the endpoint has probability 0 because X is continuous. Differentiating gives

$$f_Z(z) = -f_X(h(z))h'(z) = f_X(h(z))|h'(z)|.$$

Interpretation

- $h'(z)$ accounts for the stretching/compression of intervals.
- The absolute value ensures the density remains nonnegative.

Example: Revisit $Z = e^X$

Example

Let $X \sim \text{Uniform}(0, 1)$, and let

$$Z = e^X.$$

Here

$$z = e^x \iff x = h(z) = \log z.$$

Also,

$$h'(z) = \frac{1}{z}.$$

Since $f_X(x) = 1$ for $0 < x < 1$, we must have $0 < \log z < 1$, i.e., $1 < z < e$. Thus,

$$f_Z(z) = f_X(\log z) \left| \frac{1}{z} \right| = \frac{1}{z}, \quad 1 < z < e.$$

This agrees with the CDF-first calculation.

Example: Reciprocal transformation

Example

Let $X \sim \text{Uniform}(1, 2)$, and let

$$Z = \frac{1}{X}.$$

The transformation is strictly decreasing. Its inverse is

$$x = h(z) = \frac{1}{z}, \quad |h'(z)| = \frac{1}{z^2}.$$

Since $1 < X < 2$, the possible values of Z are $\frac{1}{2} < Z < 1$. Since $f_X(x) = 1$ for $1 < x < 2$,

$$f_Z(z) = f_X(1/z) \frac{1}{z^2} = \frac{1}{z^2}, \quad \frac{1}{2} < z < 1.$$

Thus,

$$f_Z(z) = \begin{cases} z^{-2}, & \frac{1}{2} < z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Special case: Linear transformations

Let

$$Z = aX + b, \quad a \neq 0.$$

This is a special instance of strictly monotonic transforms.

Then

$$x = h(z) = \frac{z - b}{a}, \quad |h'(z)| = \frac{1}{|a|}.$$

Thus,

$$f_Z(z) = \frac{1}{|a|} f_X\left(\frac{z - b}{a}\right).$$

Interpretation:

- b shifts the distribution.
- a stretches or compresses the scale.
- The factor $1/|a|$ preserves total area 1.

Example: Linear transformation of a uniform random variable

Example

Let $X \sim \text{Uniform}(0, 1)$, and let

$$Z = 2X + 3.$$

Then

$$h(z) = \frac{z - 3}{2}, \quad |h'(z)| = \frac{1}{2}.$$

Since $f_X(x) = 1$ for $0 < x < 1$, the possible values of Z are $3 < z < 5$, and

$$f_Z(z) = \frac{1}{2} f_X\left(\frac{z - 3}{2}\right) = \frac{1}{2}, \quad 3 < z < 5.$$

Thus,

$$Z \sim \text{Uniform}(3, 5).$$

Pop-up quiz: Transformation

Let $X \sim \text{Uniform}(0, 1)$, and let

$$Z = 2 - X.$$

Which support and density are correct for Z ?

- A) $1 < z < 2$, $f_Z(z) = 1$
- B) $0 < z < 1$, $f_Z(z) = 1$
- C) $1 < z < 2$, $f_Z(z) = 2 - z$
- D) $0 < z < 2$, $f_Z(z) = \frac{1}{2}$

Answer: A.

Here $x = 2 - z$, so $|dx/dz| = 1$. Also,

$$0 < x < 1 \iff 1 < z < 2.$$

Thus $f_Z(z) = 1$ for $1 < z < 2$.

Follow-up: Follow-up: Let $W = 3X - 2$. What is the support and PDF of W ?

Function of two random variables

Let (X, Y) be jointly continuous, and let

$$Z = h(X, Y).$$

To find the distribution of Z , a general strategy is again to identify the CDF first:

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(h(X, Y) \leq z) \\ &= \iint_{\{(x,y):h(x,y)\leq z\}} f_{X,Y}(x, y) \, dx \, dy. \end{aligned}$$

Then, when differentiable,

$$f_Z(z) = F'_Z(z).$$

Main challenge: Identifying the integration region correctly.

Practical rules: Draw the region $\{(x, y) : h(x, y) \leq z\}$. Split the integral as needed whenever the boundary changes.

Example: Product of two independent uniforms (1/2)

Example

Let (X, Y) be uniformly distributed on the unit square $[0, 1]^2$, and let

$$Z = XY.$$

Question: Find the CDF and the PDF of Z .

For $0 < z < 1$, $F_Z(z) = P(XY \leq z)$. For a fixed $x \in (0, 1)$,

$$XY \leq z \iff Y \leq \frac{z}{x}.$$

Geometrically, this is the area below the curve $y = z/x$ inside the unit square.

- If $0 < x \leq z$, then $z/x \geq 1$, so all $0 < Y < 1$ are allowed.
- If $z < x < 1$, then $0 < Y \leq z/x$.

$$F_Z(z) = \int_0^z 1 \, dx + \int_z^1 \frac{z}{x} \, dx = z + z \log(1/z), \quad 0 < z < 1.$$

Example: Product of two independent uniforms (2/2)

Example

For $Z = XY$ with $(X, Y) \sim \text{Uniform}([0, 1]^2)$, we found

$$F_Z(z) = \begin{cases} 0, & z \leq 0, \\ z + z \log(1/z), & 0 < z < 1, \\ 1, & z \geq 1. \end{cases}$$

Differentiating,

$$f_Z(z) = \frac{d}{dz} \left[z + z \log\left(\frac{1}{z}\right) \right] = \log\left(\frac{1}{z}\right), \quad 0 < z < 1.$$

Therefore,

$$f_Z(z) = \begin{cases} \log(1/z), & 0 < z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Takeaway: For $Z = h(X, Y)$, the key step is identifying the region $\{h(x, y) \leq z\}$.

Example: Ratio of two independent uniforms (1/3)

Example

Let (X, Y) be uniformly distributed on the unit square $[0, 1]^2$, and let

$$Z = \frac{Y}{X}.$$

Since $P(X = 0) = 0$, the ratio is well-defined with probability 1.

Question: Find the CDF and the PDF of Z .

For $z < 0$, $F_Z(z) = 0$, since $Z \geq 0$.

For $0 \leq z < 1$,

$$\begin{aligned} F_Z(z) &= P\left(\frac{Y}{X} \leq z\right) = P(Y \leq zX) \\ &= \int_0^1 \int_0^{zx} 1 \, dy \, dx = \int_0^1 zx \, dx = \frac{z}{2}. \end{aligned}$$

Example: Ratio of two independent uniforms (2/3)

Example

For $z \geq 1$, split according to whether $zx \leq 1$:

$$\begin{aligned}F_Z(z) &= \int_0^{1/z} \int_0^{zx} 1 \, dy \, dx + \int_{1/z}^1 \int_0^1 1 \, dy \, dx \\&= \int_0^{1/z} zx \, dx + \int_{1/z}^1 1 \, dx \\&= \frac{1}{2z} + 1 - \frac{1}{z} = 1 - \frac{1}{2z}.\end{aligned}$$

Thus,

$$F_Z(z) = \begin{cases} 0, & z < 0, \\ z/2, & 0 \leq z < 1, \\ 1 - \frac{1}{2z}, & z \geq 1. \end{cases}$$

Example: Ratio of two independent uniforms (3/3)

Example

Differentiating the CDF with respect to z ,

$$f_Z(z) = \begin{cases} \frac{1}{2}, & 0 < z < 1, \\ \frac{1}{2z^2}, & z > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Takeaways:

- A ratio can have unbounded support even when $X, Y \in (0, 1)$.
- The CDF is piecewise because the line $y = zx$ hits the top of the unit square when $z \geq 1$.
- Check:

$$\int_{\mathbb{R}} f_Z(z) dz = \int_0^1 \frac{1}{2} dz + \int_1^{\infty} \frac{1}{2z^2} dz = 1.$$

Sum of independent random variables: Convolution

Let (X, Y) be jointly continuous and let

$$Z = X + Y.$$

If X and Y are independent continuous random variables, then

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

The density of $Z = X + Y$ is obtained by the **convolution** operation:

$$f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx.$$

- **Support rule:** The integral only includes x 's for which both

$$x \in \text{supp}(X) \quad \text{and} \quad z - x \in \text{supp}(Y).$$

- The convolution of PMFs is defined similarly: $p_Z(z) = (p_X * p_Y)(z) = \sum_x p_X(x) p_Y(z - x)$.
- **Interpretation:** To get $X + Y = z$, sum over all possible ways $x + (z - x) = z$.

Example: Sum of two independent uniforms

Example

Let

$$X, Y \stackrel{\text{independent}}{\sim} \text{Uniform}(0, 1), \quad \text{and} \quad Z = X + Y.$$

Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx.$$

Since $f_X(x) = 1$ for $0 < x < 1$ and $f_Y(z-x) = 1$ when $0 < z-x < 1$, the integrand $f_X(x)f_Y(z-x)$ is nonzero only when

$$0 < x < 1 \quad \text{and} \quad z-1 < x < z.$$

Thus x must lie in

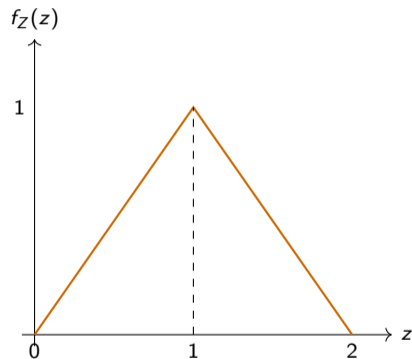
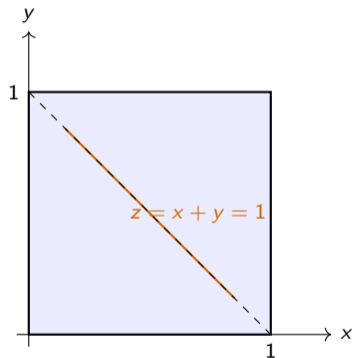
$$\max(0, z-1) < x < \min(1, z).$$

The density of Z at z is equal to the length of this interval (whenever it is positive):

$$f_Z(z) = \begin{cases} z, & 0 < z < 1, \\ 2-z, & 1 \leq z < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Geometric view of $X + Y$

For $X, Y \sim \text{Uniform}(0, 1)$, the joint density is uniform on the unit square.



For a thin band $z < X + Y < z + \Delta z$, the probability is the area of the band. As $\Delta z \rightarrow 0$, this area is proportional to the length of the slice $x + y = z$ inside the unit square.

This length increases for $0 < z < 1$ and decreases for $1 < z < 2$, producing a triangular density.

Example: Sum of independent exponentials

Example

Let X, Y be independent $\text{Exponential}(\lambda)$ random variables, and let

$$Z = X + Y.$$

Since $X, Y > 0$, we have $Z > 0$. Using convolution, for $z > 0$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x)f_Y(z-x) dx = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\ &= \lambda^2 e^{-\lambda z} \int_0^z dx = \lambda^2 z e^{-\lambda z}. \end{aligned}$$

Thus,

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & z > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note: This distribution is often called an Erlang/Gamma distribution with shape 2.

Pop-up quiz: Convolution

Let X, Y be independent $\text{Uniform}(0, 1)$, and let

$$Z = X + Y.$$

Which describes the density of Z correctly?

A) $f_Z(z) = \frac{1}{2}, \quad 0 < z < 2$

C) $f_Z(z) = 1, \quad 0 < z < 2$

B) $f_Z(z) = \begin{cases} z, & 0 < z < 1, \\ 2 - z, & 1 < z < 2, \\ 0, & \text{otherwise} \end{cases}$

D) $f_Z(z) = z, \quad 0 < z < 2$

Answer: B.

For fixed z , the contributing values of x satisfy

$$\max(0, z - 1) < x < \min(1, z).$$

The length is z for $0 < z < 1$, and $2 - z$ for $1 < z < 2$.

Wrap-up

Derived distributions: If $Z = g(X)$, a general method is:

$$F_Z(z) = P(g(X) \leq z), \quad f_Z(z) = F'_Z(z).$$

- This CDF-first method is generic, and especially useful when g is not one-to-one.
- Always identify the support of Z carefully.

Monotone transformations: If g is monotone with inverse $h = g^{-1}$, then

$$f_Z(z) = f_X(h(z))|h'(z)|.$$

Functions of two random variables: For $Z = h(X, Y)$, compute

$$F_Z(z) = P(h(X, Y) \leq z).$$

- If $Z = X + Y$ and X, Y are independent, then f_Z is the convolution of f_X and f_Y :

$$f_Z(z) = \int f_X(x)f_Y(z-x) dx.$$

Suggested reading: [BT08, Ch. 4.1]

References



Dimitri Bertsekas and John N Tsitsiklis.

Introduction to probability, volume 1.

Athena Scientific, 2nd edition, 2008.