STA 131A: Introduction to Probability Theory

Lecture 22: Sums of Independent Random Variables

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Agenda

Last time:

- We finished the law of total variance:

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(\mathbb{E}[X \mid Y])$$

- We introduced the moment generating function:

$$M_X(t) = \mathbb{E}[e^{tX}]$$

Today:

- MGFs of transformed random variables
  - Linear transformations
  - Sums of independent random variables
- Random sums: $S = X_1 + \cdots + X_N$, where N is random
  - Mean and variance by conditioning on N
  - MGF of a random sum

Recap: Moment generating functions

The **moment generating function** of X is

$$M_X(t) = \mathbb{E}[e^{tX}],$$

for values of t where this expectation is finite.

- We can compute moments from MGFs

$$M_X^{(k)}(0) = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \mathbb{E}[X^k].$$

- If $M_X(t)$ is finite on some open interval around 0, then M_X uniquely determines the distribution of X.

Today:

MGFs $\longrightarrow$ MGFs of linear transforms and sums $\longrightarrow$ random sums of RVs.

MGF of a linear transformation

MGF of a linear transformation

Let X be a random variable with MGF $M_X(t)$, and let $a, b \in \mathbb{R}$. Then

$$M_{aX+b}(t) = e^{bt} M_X(at),$$

whenever $M_X(at)$ is finite.

Proof. Let $Y = aX + b$. Then

$$\begin{aligned} M_Y(t) &= \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(aX+b)}] \\ &= e^{bt}\mathbb{E}[e^{(at)X}] \\ &= e^{bt} M_X(at). \end{aligned}$$

Interpretation:

- Adding b multiplies the MGF by e^{bt}.
- Scaling by a changes the input to the MGF from t to at.

Example: Linear transformation of a normal random variable

Example

Suppose $X \sim N(\mu, \sigma^2)$, whose MGF is

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).$$

Let $Y = aX + b$. Then

$$\begin{aligned} M_Y(t) &= e^{bt} M_X(at) \\ &= e^{bt} \exp\left(\mu a t + \frac{\sigma^2 a^2 t^2}{2}\right) \\ &= \exp\left((a\mu + b)t + \frac{a^2\sigma^2 t^2}{2}\right). \end{aligned}$$

Thus,

$$Y \sim N(a\mu + b,\ a^2\sigma^2).$$

MGF of a sum of independent random variables

MGF of a sum of independent random variables

Let X and Y be independent. Then

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

Proof.

$$\begin{aligned}
M_{X+Y}(t) &= \mathbb{E}[e^{t(X+Y)}] \\
&= \mathbb{E}[e^{tX}e^{tY}] \\
&= \mathbb{E}[e^{tX}]\,\mathbb{E}[e^{tY}] \qquad \text{(by independence)} \\
&= M_X(t)M_Y(t).
\end{aligned}$$

More generally, if $X_1, \ldots, X_n$ are independent, then

$$M_{X_1+\cdots+X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

Example: Sum of Bernoulli variables

Example

Let $X_1, \ldots, X_n$ be independent $\mathrm{Bernoulli}(p)$ random variables, and let

$$S = X_1 + \cdots + X_n.$$

For one Bernoulli random variable,

$$M_{X_i}(t) = 1 - p + pe^t.$$

Therefore,

$$M_S(t) = \prod_{i=1}^{n} M_{X_i}(t) = (1 - p + pe^t)^n.$$

This is the MGF of a $\mathrm{Binomial}(n, p)$ random variable

$$S \sim \mathrm{Binomial}(n, p).$$

Example: Sum of independent Poisson variables

Example

Let

$$X \sim \mathrm{Poisson}(\lambda_1), \qquad Y \sim \mathrm{Poisson}(\lambda_2),$$

and suppose X and Y are independent.

Recall the MGF of $\mathrm{Poisson}(\lambda)$ is

$$M(t) = \exp\left[\,\lambda(e^t - 1)\,\right].$$

Therefore,

$$\begin{aligned} M_{X+Y}(t) &= M_X(t)M_Y(t) \\ &= \exp\left[\,\lambda_1(e^t - 1)\,\right]\exp\left[\,\lambda_2(e^t - 1)\,\right] \\ &= \exp\left[\,(\lambda_1 + \lambda_2)(e^t - 1)\,\right]. \end{aligned}$$

Thus,

$$X + Y \sim \mathrm{Poisson}(\lambda_1 + \lambda_2).$$

Example: Sum of independent normal variables

Example

Suppose $X_1, \ldots, X_n$ are independent, and let

$$S = \sum_{i=1}^{n} X_i \qquad \text{where} \qquad X_i \sim N(\mu_i, \sigma_i^2), \qquad i = 1, \ldots, n,$$

Recall that $M_Z(t) = e^{t^2/2}$ when $Z \sim N(0,1)$. Thus, if $X = \sigma Z + \mu$, then

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right),$$

and therefore, we have

$$M_S(t) = \prod_{i=1}^{n} M_{X_i}(t) = \exp\left[\left(\sum_{i=1}^{n} \mu_i\right) t + \frac{\left(\sum_{i=1}^{n} \sigma_i^2\right) t^2}{2}\right].$$

By uniqueness of MGFs (inversion property), $S \sim N\left(\sum_{i=1}^{n} \mu_i,\ \sum_{i=1}^{n} \sigma_i^2\right)$.

Pop-up quiz: MGFs and sums

Suppose X and Y are independent, with

$$M_X(t) = \frac{1}{2} + \frac{1}{2}e^t, \qquad M_Y(t) = \exp\{3(e^t - 1)\}.$$

Question: What is $M_{X+Y}(t)$?

A) $\frac{1}{2} + \frac{1}{2}e^t + \exp\{3(e^t - 1)\}$

B) $\left(\frac{1}{2} + \frac{1}{2}e^t\right) \exp\{3(e^t - 1)\}$

C) $\exp\{(3.5)(e^t - 1)\}$

D) Cannot be determined from the given information

Answer: B. Because X and Y are independent,

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \left(\frac{1}{2} + \frac{1}{2}e^t\right) \exp\{3(e^t - 1)\}.$$

Why random sums?

So far, sums usually had a fixed number of terms:

$$X_1 + \cdots + X_n.$$

But in many applications, the number of terms is itself random:

$$S = X_1 + \cdots + X_N.$$

Examples:

- Insurance: total payout = sum of a random number of claims
- Queueing: total workload = sum of service times for a random number of customers
- Revenue: total revenue = sum of purchase amounts from a random number of purchases
- Thinning: number of successes among a random number of trials

Question: How do we compute $\mathbb{E}[S]$, $\mathrm{Var}(S)$, and $M_S(t)$?

Sum of a random number of independent random variables

Sum of a random number of variables

Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with

$$\mathbb{E}[X_i] = \mu, \qquad \mathrm{Var}(X_i) = \sigma^2.$$

Let $N \in \{0, 1, 2, \ldots\}$ be a nonnegative integer-valued random variable, independent of $X_1, X_2, \ldots$. Define

$$S = \sum_{i=1}^{N} X_i,$$

with the convention that $S = 0$ when $N = 0$. Then

$$\begin{aligned} \mathbb{E}[S] &= \mathbb{E}[N]\,\mu, \\ \mathrm{Var}(S) &= \mathbb{E}[N]\,\sigma^2 + \mu^2 \mathrm{Var}(N), \\ M_S(t) &= M_N\big(\log M_X(t)\big). \end{aligned}$$

Proof: Mean of a random sum

The key idea is to condition on N.

Given $N = n$,

$$S = X_1 + \cdots + X_n.$$

Thus,

$$\mathbb{E}[S \mid N = n] = n\mu,$$

and as a random variable,

$$\mathbb{E}[S \mid N] = N\mu.$$

Using the law of total expectation,

$$\begin{aligned} \mathbb{E}[S] &= \mathbb{E}[\mathbb{E}[S \mid N]] \\ &= \mathbb{E}[N\mu] \\ &= \mathbb{E}[N]\,\mu. \end{aligned}$$

$$\text{expected total} = \text{expected number of terms} \times \text{expected contribution per term}.$$

Proof: Variance of a random sum

Conditioned on $N = n$, S is a sum of n independent terms, so $\mathrm{Var}(S \mid N = n) = n\sigma^2$.

Hence,

$$\mathbb{E}[S \mid N] = N\mu, \qquad \mathrm{Var}(S \mid N) = N\sigma^2.$$

Using the law of total variance, we get

$$\begin{aligned}\mathrm{Var}(S) &= \mathbb{E}[\mathrm{Var}(S \mid N)] + \mathrm{Var}(\mathbb{E}[S \mid N]) \\ &= \mathbb{E}[N\sigma^2] + \mathrm{Var}(N\mu) \\ &= \sigma^2\mathbb{E}[N] + \mu^2\mathrm{Var}(N).\end{aligned}$$

Thus,

$$\mathrm{Var}(S) = \underbrace{\mathbb{E}[N]\mathrm{Var}(X)}_{\text{variation from random terms}} + \underbrace{(\mathbb{E}[X])^2\mathrm{Var}(N)}_{\text{variation from random count}}.$$

- Even if N were fixed, the terms $X_1, \ldots, X_N$ are random
- Even if each term were close to its mean, the number of terms N is random

Proof: MGF of a random sum

Given $N = n$, $S = X_1 + \cdots + X_n$, so by independence of the X_i's,

$$M_{S|N=n}(t) = \mathbb{E}[e^{tS} \mid N = n] = M_X(t)^n.$$

Equivalently,

$$\mathbb{E}[e^{tS} \mid N] = M_X(t)^N.$$

Now take expectation over N:

$$M_S(t) = \mathbb{E}\left[\, e^{tS} \,\right] = \mathbb{E}\left[\, \mathbb{E}[e^{tS} \mid N] \,\right] = \mathbb{E}\left[M_X(t)^N \right].$$

Since

$$M_X(t)^N = e^{N \log M_X(t)},$$

we obtain

$$\begin{aligned} M_S(t) &= \mathbb{E}\left[e^{N \log M_X(t)}\right] \\ &= M_N\left(\log M_X(t)\right). \end{aligned}$$

* *Note:* This formula is valid for values of t for which the expectations are finite.

Example: Compound Poisson total cost

Example

Let N be the random number of claims, and let $X_1, X_2, \dots$ be the claim sizes. Assume all variables are independent, with

$$N \sim \mathrm{Poisson}(\lambda), \qquad \mathbb{E}[X_i] = \mu, \qquad \mathrm{Var}(X_i) = \sigma^2.$$

Let

$$S = \sum_{i=1}^{N} X_i$$

be the total cost. Using the random-sum formulas,

$$\mathbb{E}[S] = \lambda\mu, \qquad \mathrm{Var}(S) = \lambda\sigma^2 + \lambda\mu^2 = \lambda\mathbb{E}[X_i^2].$$

This is a compound model: a random number of random-sized contributions.

Interpretation of variance: total cost varies because claim sizes are random and because the number of claims is random.

Example: Poisson thinning

Example

Let $N \sim \mathrm{Poisson}(\lambda)$. Given N, suppose each of the N events is marked as a success independently with probability p, independently of all other events. Let S denote the number of successes, i.e.,

$$S = \sum_{i=1}^{N} X_i, \qquad X_i \sim \mathrm{Bernoulli}(p).$$

We have

$$M_N(t) = \exp\left[\,\lambda(e^t - 1)\,\right], \qquad M_X(t) = 1 - p + pe^t.$$

Therefore,

$$\begin{aligned} M_S(t) &= M_N\big(\log M_X(t)\big) \\ &= \exp\big[\lambda(M_X(t) - 1)\big] \\ &= \exp\big[\lambda p(e^t - 1)\big]. \end{aligned}$$

This is the MGF of $\mathrm{Poisson}(\lambda p)$. Hence, $S \sim \mathrm{Poisson}(\lambda p)$.

Interpretation: randomly keeping each event with probability p thins the Poisson rate $\lambda \to \lambda p$.

Pop-up quiz: Random sums

Let $S = \sum_{i=1}^{N} X_i$, where N is independent of the i.i.d. X_i's. Suppose

$$\mathbb{E}[N] = 5, \qquad \mathrm{Var}(N) = 2, \qquad \mathbb{E}[X_i] = 3, \qquad \mathrm{Var}(X_i) = 4.$$

Question: What are $\mathbb{E}[S]$ and $\mathrm{Var}(S)$?

A) $\mathbb{E}[S] = 15, \quad \mathrm{Var}(S) = 20$
B) $\mathbb{E}[S] = 15, \quad \mathrm{Var}(S) = 38$
C) $\mathbb{E}[S] = 8, \quad \mathrm{Var}(S) = 20$
D) $\mathbb{E}[S] = 15, \quad \mathrm{Var}(S) = 18$

Answer: B.

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X] = 5 \cdot 3 = 15,$$
$$\mathrm{Var}(S) = \mathbb{E}[N]\mathrm{Var}(X) + (\mathbb{E}[X])^2 \mathrm{Var}(N) = 5 \cdot 4 + 3^2 \cdot 2 = 38.$$

Follow-up: Which term comes from the randomness of the count N?

Wrap-up

MGFs for sums of independent RVs: If X and Y are independent, then

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

- This converts convolution into multiplication.

Random sums: if N is independent of i.i.d. X_i's and $S = \sum_{i=1}^{N} X_i$, then conditioning on N gives

$$\begin{aligned}
\mathbb{E}[S] &= \mathbb{E}[N]\mathbb{E}[X], \\
\mathrm{Var}(S) &= \mathbb{E}[N]\mathrm{Var}(X) + (\mathbb{E}[X])^2\mathrm{Var}(N), \\
M_S(t) &= M_N\left(\log M_X(t)\right).
\end{aligned}$$

- **Takeaway:** random sums have two sources of variability: the random summands and the random number of summands.

Suggested reading: [BT08, Ch. 4.4 & 4.5]

References

Dimitri Bertsekas and John N Tsitsiklis.
Introduction to probability, volume 1.
Athena Scientific, 2nd edition, 2008.