STA 35C: Statistical Data Science III

Lecture 3: Bayes' Theorem and Random Variables

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Spring 2026, UC Davis

Announcements

Homework 1 is now posted

- Due: Tue, April 7, 11:59 PM PT
- Please follow the submission instructions (e.g., formatting)
- Make sure you can access the homework PDF and know where to submit your solutions
  - Late submissions will not be accepted for any reason and will receive 0 points

If you are unsure about anything related to the course:

1. See the syllabus for information
2. Check Piazza first; someone may already have asked a similar question
3. Feel free to ask on Piazza, in class, or during office hours
   - Reserve email for urgent or private matters only

Agenda

Last time:

- Set theory basics
- Probabilistic models
- Conditional probability & Independence

Today:

- Bayes' theorem
- Random variables
  - PMFs, PDFs, and CDFs
  - Expectation and variance
  - Joint, marginal, and conditional distributions
  - Covariance and correlation
  - Sums of random variables

Recap: Outcomes, events, probability

Outcome: a possible result of an experiment (often denoted ω)

- "Head" or "Tail"
- The outcome 6 on a die roll
- Stock price change of \$5

Event: a subset $A \subseteq \Omega$; the event A occurs if the realized outcome ω lies in A

- $\emptyset, \{6\}, \{1, 2, 3, 4, 5, 6\}$
- For example, if $A = \{1, 2, 3\}$, then A occurs if and only if $\omega \in A$, i.e., iff $\omega = 1, 2$, or 3
  - This does **NOT** mean "$\omega = 1$ and $\omega = 2$ and $\omega = 3$"

Probability law: a function P that assigns each event A a number $P(A)$ such that

- $P(A) \geq 0$
- $A \cap B = \emptyset \implies P(A \cup B) = P(A) + P(B)$
- $P(\Omega) = 1$

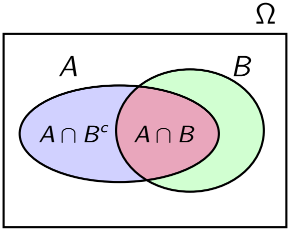
Recap: Conditional probability

For an event B with $P(B) > 0$, the **conditional probability**[1] of A given B is

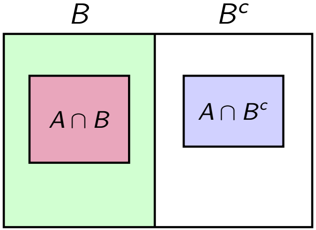
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- **Example:** $A = \{6\}$, $B = \{2, 4, 6\}$ $\longrightarrow$ $P(A) = 1/6$ vs. $P(A \mid B) = 1/3$
- **Multiplication rule:** $P(A \cap B) = P(B)P(A \mid B)$
- **Law of total probability:** If $0 < P(B) < 1$, then

$$P(A) = P(A \cap B) + P(A \cap B^c) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c)$$



partition by B
$\longrightarrow$



[1]For any fixed B, $P(\cdot \mid B)$ satisfies the three axioms

Bayes' theorem

Often, we know $P(B \mid A)$, but want $P(A \mid B)$

- A: a hypothesis/model/state of the world
- B: observed data or evidence
- Example: $A =$ having a disease, $\quad B =$ positive screening result

After observing B, Bayes' theorem updates $P(A)$ to $P(A \mid B)$

- *Prior:* $P(A)$ and $P(A^c)$
- *Likelihood:* $P(B \mid A)$ and $P(B \mid A^c)$
- *Posterior:* $P(A \mid B)$ and $P(A^c \mid B)$

Bayes' theorem states that

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)} \qquad \text{where} \qquad P(B) = P(B \mid A)P(A) + P(B \mid A^c)P(A^c)$$

- *Posterior* = *Likelihood* × *Prior* / *Evidence*

Bayes' theorem: Example 1

Example (Disease screening)

Let A = disease and B = positive screening result

- Suppose $P(A) = 0.01$
- $P(B \mid A) = 0.8$ (true positive)
- $P(B \mid A^c) = 0.1$ (false positive; positive screening though a person does not have cancer)

Q: If the test is positive, how likely is it that the person actually has the disease?

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A)\, P(A)}{P(B \mid A)\, P(A) + P(B \mid A^c)\, P(A^c)} \\ &= \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.1 \times 0.99} \approx 0.075 \end{aligned}$$

- A positive result raises the probability from 1% to about 7.5%, but it is still far from certain
- When the disease is rare, even a fairly accurate test can produce many false positives

Bayes' theorem: Example 2

Example (Fair or double-headed?)

Let A = the coin is double-headed and A^c = the coin is fair

Let B = the first toss is Head

- Suppose $P(A) = 0.5$
- $P(B \mid A) = 1$ (the double-headed (biased) coin always lands Head)
- $P(B \mid A^c) = 0.5$ (the fair coin lands Head with probability $1/2$)

Q: After observing one Head, how likely is it that the coin is double-headed?

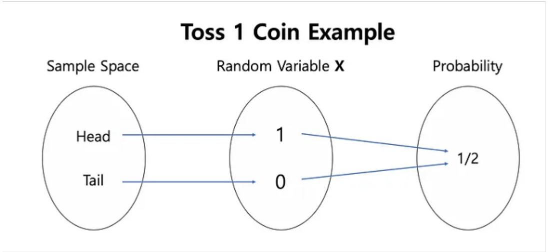
$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} \\ &= \frac{1 \times 0.5}{1 \times 0.5 + 0.5 \times 0.5} = \frac{2}{3} \approx 0.667. \end{aligned}$$

- One Head raises the probability from 50% to 66.7%
- Data increase the probability of the model that better explains the observation

Random variables

A **random variable**[2] $X : \Omega \to \mathbb{R}$ assigns a real number to each outcome $\omega \in \Omega$

- Once the outcome ω is realized, $X = X(\omega)$ is just a number
- Probabilities on Ω induce a distribution on the values of X



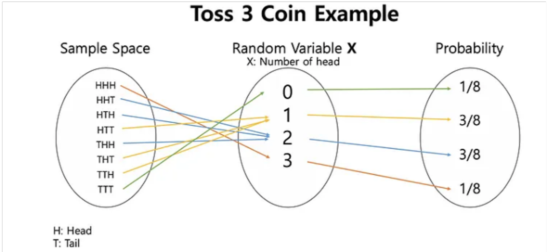


Figure: Random variable example: outcome of tossing a coin[3]

[2] A fully rigorous definition is beyond the scope of STA 35C

[3] Source: https://medium.com/jun94-devpblog/prob-stats-1-random-variable-483c45242b3c

Distribution of a random variable

How do we describe how probability is distributed over the values of a random variable?

- If X is **discrete**, its **probability mass function** (PMF) is

$$p_X(x) = P(X = x) \qquad \text{where} \qquad p_X(x) \geq 0 \quad \text{and} \quad \sum_x p_X(x) = 1$$

- If X is **continuous**, its **probability density function** (PDF) is a function $f_X \geq 0$ such that

$$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx \qquad \text{and therefore} \qquad \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

  - If X is continuous, then $P(X = x) = 0$ for every single x
  - Thus, $f_X(x)$ is a density, not a point probability

- In both cases, the **cumulative distribution function** (CDF) is

$$F_X(x) := P(X \leq x)$$

Distribution functions: Examples of discrete RVs

Example (Biased coin)

Suppose a biased coin lands Head twice as often as Tail, and define

$$X = \mathbf{1}\{\text{Head}\} = \begin{cases} 1 & \text{Head}, \\ 0 & \text{Tail}. \end{cases} \qquad \rightarrow \qquad p_X(k) = \begin{cases} 2/3 & k = 1, \\ 1/3 & k = 0, \\ 0 & \text{otherwise} \end{cases}$$

The CDF is

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1/3, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

Exercise: Let X denote the number on the face after rolling a fair die:

$$p_X(k) = \frac{1}{6}, \qquad \forall k \in \{1, 2, 3, 4, 5, 6\}$$

Write down the CDF of X and draw its graph

Distribution functions: Examples of continuous RVs

Example (Uniform)

Let X be a random variable uniform on the interval $[0, 2]$

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

For $a, b \in \mathbb{R}$ such that[a] $0 \leq a \leq b \leq 2$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx = \frac{b-a}{2}$$

The CDF

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt = \begin{cases} 0 & x < 0, \\ \frac{x}{2} & 0 \leq x < 2, \\ 1 & x \geq 2. \end{cases}$$

[a] **Question:** What if $a < 0$ or $b > 2$?

Expectation and variance

The distribution of a random variable contains all information we need

Describing the full distribution can be complicated

$\rightarrow$ We often summarize it by location and spread

- The **expected value** of a random variable X is its probability-weighted average:
  - Discrete: $\mathbb{E}[X] = \sum_x x \cdot p_X(x)$
  - Continuous: $\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$
- The **variance** of a random variable X is the “spread” around the mean:
  - $\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$
  - Alternative formula: $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Expectation and variance: Example 1

Example (Tossing a coin with head probability p)

The PMF of a Bernoulli random variable $X \sim \mathrm{Bern}(p)$ is

$$p_X(x) = p(X = x) = \begin{cases} p & \text{if } x = 1 \text{ (head)}, \\ 1 - p & \text{if } x = 0 \text{ (tail)}. \end{cases}$$

- *Expectation:*

$$\begin{aligned} \mathbb{E}[X] &= \sum_x x \cdot p(x) \\ &= 0 \cdot (1 - p) + 1 \cdot p \\ &= p \end{aligned}$$

- *Variance:*

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \sum_x (x - p)^2 \cdot p(x) \\ &= (-p)^2 \cdot (1 - p) + (1 - p)^2 \cdot p \\ &= p(1 - p) \end{aligned}$$

Expectation and variance: Example 2

Example (Uniform distribution)

The PDF of a uniform random variable on $[a, b]$ is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

Expectation:

$$\begin{aligned} \mathbb{E}[X] &= \int_a^b x \cdot \frac{1}{b-a}\, dx \\ &= \frac{a+b}{2} \end{aligned}$$

Variance:

$$\begin{aligned} \mathbb{E}[X^2] &= \int_a^b x^2 \cdot \frac{1}{b-a}\, dx \\ &= \frac{a^2+ab+b^2}{3} \\ \mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

Some properties of expectation and variance

Let X be a random variable and $a, b \in \mathbb{R}$

Expectation:

- $\mathbb{E}[a] = a$

$$\because \; \mathbb{E}[a] = \sum_x a\, p_X(x) = a \underbrace{\sum_x p_X(x)}_{=1} = a$$

- $\mathbb{E}[bX] = b \cdot \mathbb{E}[X]$

$$\because \; \mathbb{E}[bX] = \sum_x (bx)\, p_X(x) = b \sum_x x\, p_X(x) = b\, \mathbb{E}[X]$$

Variance:

- $\mathrm{Var}(a) = 0$
- $\mathrm{Var}(bX) = b^2 \cdot \mathrm{Var}(X)$

$$\because \; \mathrm{Var}(bX) = \mathbb{E}[(bX)^2] - \mathbb{E}[bX]^2 = b^2 \cdot \mathbb{E}[X^2] - \big(b \cdot \mathbb{E}[X]\big)^2 = b^2\, \mathrm{Var}(X)$$

Distribution of multiple random variables

Let X, Y be discrete random variables

Joint distribution:

- The joint PMF of X and Y satisfies $p_{X,Y}(x, y) = P(X = x,\ Y = y)$
- The joint CDF of X and Y is defined by $F_{X,Y}(x, y) = P(X \leq x,\ Y \leq y)$

Marginal distributions: The marginal PMF of X (and of Y) are:

$$p_X(x) = \sum_y p_{X,Y}(x, y), \qquad p_Y(y) = \sum_x p_{X,Y}(x, y)$$

Conditional distribution: The conditional PMF of X given $Y = y$ is

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x, y)}{p_Y(y)} = \frac{p_{X,Y}(x, y)}{\sum_{x'} p_{X,Y}(x', y)}$$

- **Independence:** X and Y are independent if $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$ for all x, y

For continuous RVs, PMFs are replaced by PDFs and sums are replaced by integrals

Joint, marginal, conditional distributions: Example

Example (One fair die)

Suppose rolling a fair die and let

$$X = \mathbf{1}\{\text{roll is even}\}, \qquad Y = \mathbf{1}\{\text{roll} > 3\}.$$

$$(X, Y) = \begin{cases} (0,0) & \text{if } \omega \in \{1,3\} \\ (0,1) & \text{if } \omega \in \{5\} \\ (1,0) & \text{if } \omega \in \{2\} \\ (1,1) & \text{if } \omega \in \{4,6\} \end{cases} \implies$$

	$Y=0$	$Y=1$	$p_X(x)$
$X=0$	$1/3$	$1/6$	$1/2$
$X=1$	$1/6$	$1/3$	$1/2$
$p_Y(y)$	$1/2$	$1/2$	

So the conditional probability

$$p_{X\mid Y}(1 \mid 1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{1/3}{1/2} = \frac{2}{3}$$

Also, X and Y are not independent, since

$$p_{X,Y}(1,1) = \frac{1}{3} \neq p_X(1)p_Y(1) = \frac{1}{4}$$

Covariance and correlation

The **covariance** between X and Y measures how they vary together:

$$\mathrm{Cov}(X, Y) = \mathbb{E}\big[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\big] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

- $\mathrm{Cov}(X, Y) > 0$: large values of X tend to occur with large values of Y
- $\mathrm{Cov}(X, Y) < 0$: large values of X tend to occur with small values of Y
- If X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$; the converse is false in general

Correlation is the normalized version of covariance:

$$\rho(X, Y) := \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} \in [-1, 1] \qquad \text{(when } \mathrm{Var}(X), \mathrm{Var}(Y) > 0\text{).}$$

Example (One fair die)

	$Y = 0$	$Y = 1$
$X = 0$	$1/3$	$1/6$
$X = 1$	$1/6$	$1/3$

- $\mathrm{Cov}(X, Y) = \frac{1}{12}$
- $\rho(X, Y) = \frac{1}{3}$

Sum of random variables: Expectation and variance are easy

Sums arise naturally when we combine several sources of randomness:

- Total number of heads in n tosses: $S_n = X_1 + \cdots + X_n$
- Total score, total revenue, total waiting time
- Total measurement error from several noisy components

For $Z = X + Y$, the expectation and variance are often easy to compute:

- $\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y]$

$$\because \ \mathbb{E}[X+Y] = \sum_{x,y}(x+y)\,p_{X,Y}(x,y) = \sum_{x} x\,p_X(x) + \sum_{y} y\,p_Y(y)$$
$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

- $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$
  - *Exercise:* Verify this using properties of expectation
  - If X and Y are independent, then $\mathrm{Cov}(X,Y) = 0$, so $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Sum of random variables: Distributions can be harder

Even when $\mathbb{E}[X+Y]$ and $\mathrm{Var}(X+Y)$ are easy, the full distribution of $X+Y$ may not be

- We usually need the *joint distribution* of (X,Y), not just the separate marginals

Example (Same marginals, different joint)

Suppose X_1 and X_2 each take values 0 or 1 with probability $1/2$

- If X_1 and X_2 are independent, then

$$P(X_1+X_2=k)=\begin{cases}1/4 & k=0,\\ 1/2 & k=1,\\ 1/4 & k=2\end{cases}$$

- If $X_1=X_2$ always, then

$$P(X_1+X_2=k)=\begin{cases}1/2 & k=0,\\ 1/2 & k=2\end{cases}$$

Wrap-up

- Bayes' theorem updates prior beliefs to posterior beliefs based on observed data
- A random variable is a numerical function $X : \Omega \to \mathbb{R}$
  - Probabilities on Ω induce a distribution on the values of X
- PMFs, PDFs, and CDFs describe distributions
- Expectation and variance summarize center and spread, and facilitates computation
- Joint, marginal, and conditional distributions
  - Covariance and correlation summarize linear association
- For sums of random variables, expectations and variances are often easy to compute